Journal of Engineering and Science Research 4 (6): 01-07, 2020
e-ISSN: 2289-7127

DOI: 10.26666/rmp.jesr.2020.6.1

Symmetry Reduction of Rayleigh Wave Equation

Joseph Boon Zik Hong, Cheong Tau Han, Teoh Sian Hoon
Department of Science and Mathematics, Faculty of Eduction, UiTM, Puncak Alam, Selangor, Malaysia
josephboon@uitm.edu.my

Abstract: Rayleigh Wave Equation are significantly important in describes the speed of the wave propagating on the free surface of an isotropic half-space and it is often used as a model for physical phenomena such as large amplitude vibrations of wind-blown, ice-laden power transmission lines. The equation also appears in solution of Lamb's problem for an impulsive force. Because of its significance, Lie symmetry reduction were chosen to reduce the equation and hence solve the equation. In this paper, calculation of symmetry of the equation was first present, followed by reduction of the equation and hence analytic solution of the partial differential equation was obtained.

Keywords—*Rayleigh Wave Equation; Lie symmetry; ; partial differential equation*

I. INTRODUCTION

In the modern society of today, differential equation is applied extensively in many fields which includes mathematics, physics, biology, engineering as well as economics. It is commonly used to model real world problems that are constantly changing since derivatives describe the rate of change mathematically. The nonlinear partial differential equation has been used in a variety of physical theories including dynamics to generate canonical transformation, continuum mechanics to record conservation of mass, momentum and so on as well as optics to describe wave fronts [1]. This project will focus on a nonlinear partial differential equation which is the Rayleigh wave equation,

$$u_{\xi\xi} - u_{\varsigma\varsigma} = \varepsilon\left(u_\xi - {u_\xi}^3\right) \quad (1)$$

where ε is an arbitrary constant. The equation describes the speed of the wave propagating on the free surface of an isotropic half-space and it is often used as a model for physical phenomena such as large amplitude vibrations of wind-blown, ice-laden power transmission lines, etc [1]. In 1978, William S. Hall studied 'The Rayleigh wave equation – an analysis' to analyze the Rayleigh wave equation using two-timing method [2]. The analysis first described the system's evolution from a given initial condition on long finite intervals, and then proved the existence, determined the approximate form and the stability of the infinitely many 2π-periodic steady states [2]. The result of the analysis shows that Rayleigh wave equation has infinitely many exponentially asymptotically stable 2π-periodic solutions which only mildly satisfy the equation and that indicates that they are standing triangular waves that have at least one and possibly infinite sharp bends [2]. Therefore, he concluded that Rayleigh wave equation is not a good model for galloping oscillations [2].

One of the most recent studies is 'Successive approximation method for Rayleigh wave equation' by Saad A. Manaa, Fadhil H. Easif and Omar A. Abdulkareem in the year 2015 to find an approximate solution numerically by using successive approximation method and finite difference method [1]. Finite difference method is first carried out by approximating and replacing the equation's derivatives with Taylor's expansion, changing the equation from differential to algebraic by substituting the algebraic equations found for u_ξ, $u_{\xi\xi}$ and $u_{\varsigma\varsigma}$ into the Rayleigh wave equation and then solving [1]. Meanwhile, successive approximation method is performed by using

$$u_n(\xi,\varsigma) = u_0(\xi,0) + \iint_0^t \frac{\partial^2 u_{n-1}(\xi,s)}{\partial s^2} dsds + \varepsilon \iint_0^t \frac{\partial u_{n-1}(\xi,s)}{\partial s} dsds + \varepsilon \iint_0^t \frac{\partial u_{n-1}(\xi,s)^3}{\partial s} dsds$$

as an approximation to a solution where $n = 1$ is first substituted into the function to get the first approximation [1]. The first approximation is then used to find the second approximation for a better result, and the process will be continuing for other values of n until the resulting function matches the given equation when it is substituted with the function [1]. Both methods are applied to obtain the numerical result of the equation and it is found that successive approximation method is faster, easier and more accurate as compared to finite difference method in solving this type of problem [1]. Compare to the past research, purpose of this study is to solve the Equation (1) analytically.

Corresponding Author: Joseph Boon Zik Hong, Department of Science and Mathematics, Faculty of Eduction, UiTM, Puncak Alam, Selangor, Malaysia. Email: josephboon@uitm.edu.my

II. SYMMETRY OF RAYLEIGH WAVE EQUATION

Symmetry analysis is one of useful method to deal with partial differential equation (PDE) and ordinary differential equation (ODE) [3]. This can be seen from the past research regarding applied symmetry analysis to solve differential equation [4, 5]. In order apply symmetry analysis, one have to find symmetry of the equation. The general infinitesimal operator (symmetry) [6] is defined as

$$G = \varphi\frac{\partial}{\partial\varsigma} + \tau\frac{\partial}{\partial\xi} + \phi\frac{\partial}{\partial u}.$$

The derivation of first order prolongation of infinitesimal operator

$$G^{[1]} = \varphi\frac{\partial}{\partial\varsigma} + \tau\frac{\partial}{\partial\xi} + \phi\frac{\partial}{\partial u} + \phi^{\xi}\frac{\partial}{\partial u^{\xi}} + \phi^{\varsigma}\frac{\partial}{\partial u^{\varsigma}}$$

whereas the second order prolongation is

$$G^{[2]} = \varphi\frac{\partial}{\partial\varsigma} + \tau\frac{\partial}{\partial\xi} + \phi\frac{\partial}{\partial u} + \phi^{\xi}\frac{\partial}{\partial u^{\xi}} + \phi^{\varsigma}\frac{\partial}{\partial u^{\varsigma}} + \phi^{\xi\xi}\frac{\partial}{\partial u^{\xi\xi}} + \phi^{\xi\varsigma}\frac{\partial}{\partial u^{\xi\varsigma}} + \phi^{\varsigma\varsigma}\frac{\partial}{\partial u^{\varsigma\varsigma}}$$

with coefficients

$$\begin{aligned}
\phi^{\xi} &= D_{\xi}(\phi) - u_{\varsigma}D_{\xi}(\varphi) - u_{\xi}D_{\xi}(\tau),\\
\phi^{\varsigma} &= D_{\varsigma}(\phi) - u_{\xi}D_{\varsigma}(\varphi) - u_{\xi}D_{\varsigma}(\tau),\\
\phi^{\xi\xi} &= D_{\xi}(\phi^{\xi}) - u_{\xi\varsigma}D_{\xi}(\varphi) - u_{\xi\xi}D_{\xi}(\tau),\\
\phi^{\xi\varsigma} &= D_{\varsigma}(\phi^{\xi}) - u_{\xi\varsigma}D_{\varsigma}(\varphi) - u_{\xi\xi}D_{\varsigma}(\tau),\\
\phi^{\varsigma\varsigma} &= D_{\varsigma}(\phi^{x}) - u_{\varsigma\varsigma}D_{\varsigma}(\varphi) - u_{\xi\varsigma}D_{\varsigma}(\tau).
\end{aligned}$$

In order to calculate symmetry of equation (1), applied

$$G^{[2]}\left(u_{\xi\xi} - u_{\varsigma\varsigma} - au_{\xi} + au_{\xi}{}^{3}\right) = 0. \qquad (2)$$

After simplified, one can get equation (2) as

$$-a\phi^{\xi} + 3au_{\xi}{}^{2}\phi^{\xi} + \phi^{\xi\xi} - \phi^{\varsigma\varsigma} = 0\,. \qquad (3)$$

Since ϕ^{ξ}, $\phi^{\varsigma\varsigma}$ and $\phi^{\xi\xi}$ are present, these three terms are to be derived as follow:

$$\phi^{\xi} = \phi_{\xi} + \phi_u u_{\xi} - \varphi_{\xi}u_{\varsigma} - \varphi_u u_{\varsigma}u_{\xi} - \tau_{\xi}u_{\xi} - \tau_u u_{\xi}{}^{2}$$

$$\begin{aligned}
\phi^{\xi\xi} = {} & \phi_{\xi\xi} + 2\phi_{\xi u}u_{\xi} - \tau_{\xi\xi}u_{\xi} - \varphi_{\xi\xi}u_{\varsigma} + \phi_{uu}u_{\xi}{}^{2} - 2\tau_{\xi u}u_{\xi}{}^{2} - 2\varphi_{\xi u}u_{\varsigma}u_{\xi} - \tau_{uu}u_{\xi}{}^{3} - \varphi_{uu}u_{\varsigma}u_{\xi}{}^{2} + \phi_u u_{\xi\xi} - 2\tau_{\xi}u_{\xi\xi} - 2\varphi_{\xi}u_{\varsigma\xi}\\
& - 3\tau_u u_{\xi}u_{\xi\xi} - \varphi_u u_x u_{\xi\xi} - 2\varphi_u u_{\xi}u_{\varsigma\xi}
\end{aligned}$$

$$\begin{aligned}
\phi^{\varsigma\varsigma} = {} & \phi_{\varsigma\varsigma} + 2\phi_{\varsigma u}u_{\varsigma} - \varphi_{\varsigma\varsigma}u_{\varsigma} - \tau_{\varsigma\varsigma}u_{\xi} + \phi_{uu}u_{\varsigma}{}^{2} - 2\varphi_{\varsigma u}u_{\varsigma}{}^{2} - 2\tau_{\varsigma u}u_{\varsigma}u_{\xi} - \varphi_{uu}u_{\varsigma}{}^{3} - \tau_{uu}u_{\varsigma}{}^{2}u_{\xi} + \phi_u u_{\varsigma\varsigma} - 2\varphi_{\varsigma}u_{\varsigma\varsigma} - 2\tau_{\varsigma}u_{\varsigma\xi}\\
& - 3\varphi_u u_{\varsigma}u_{\varsigma\varsigma} - \tau_u u_{\xi}u_{\varsigma\varsigma} - 2\tau_u u_{\varsigma}u_{\xi\varsigma}
\end{aligned}$$

Substitue ϕ^{ξ}, $\phi^{\varsigma\varsigma}$, $\phi^{\xi\xi}$ and equation (1) into equation (3),

$$\begin{aligned}
0 = {} & -a\phi_{\xi} - a\phi_u u_{\xi} + a\varphi_{\xi}u_{\varsigma} + a\varphi_u u_{\varsigma}u_{\xi} + a\tau_{\xi}u_{\xi} + a\tau_u u_{\xi}{}^{2} + 3a\phi_{\xi}u_{\xi}{}^{2} + 3a\phi_u u_{\xi}{}^{3} - 3a\varphi_{\xi}u_{\varsigma}u_{\xi}{}^{2} - 3a\varphi_u u_{\varsigma}u_{\xi}{}^{3} - 3a\tau_{\xi}u_{\xi}{}^{3}\\
& - 3a\tau_u u_{\xi}{}^{4} + \phi_{\xi\xi} + 2\phi_{\xi u}u_{\xi} - \tau_{\xi\xi}u_{\xi} - \varphi_{\xi\xi}u_{\varsigma} + \phi_{uu}u_{\xi}{}^{2} - 2\tau_{\xi u}u_{\xi}{}^{2} - 2\varphi_{\xi u}u_{\varsigma}u_{\xi} - \tau_{uu}u_{\xi}{}^{3} - \varphi_{uu}u_{\varsigma}u_{\xi}{}^{2}\\
& + \phi_u\left(u_{\varsigma\varsigma} + au_{\xi} - au_{\xi}{}^{3}\right) - 2\tau_{\xi}\left(u_{\varsigma\varsigma} + au_{\xi} - au_{\xi}{}^{3}\right) - 2\varphi_{\xi}u_{\varsigma\xi} - 3\tau_u u_{\xi}\left(u_{\varsigma\varsigma} + au_{\xi} - au_{\xi}{}^{3}\right)\\
& - \varphi_u u_{\varsigma}\left(u_{\varsigma\varsigma} + au_{\xi} - au_{\xi}{}^{3}\right) - 2\varphi_u u_{\xi}u_{\xi\varsigma} - \phi_{\varsigma\varsigma} - 2\phi_{\varsigma u}u_{\varsigma} + \varphi_{\varsigma\varsigma}u_{\varsigma} + \tau_{\varsigma\varsigma}u_{\xi} - \phi_{uu}u_{\varsigma}{}^{2} + 2\varphi_{\varsigma u}u_{\varsigma}{}^{2} + 2\tau_{\varsigma u}u_{\varsigma}u_{\xi}\\
& + \varphi_{uu}u_{\varsigma}{}^{3} + \tau_{uu}u_{\varsigma}{}^{2}u_{\xi} - \phi_u u_{\varsigma\varsigma} + 2\varphi_x u_{\varsigma\varsigma} + 2\tau_{\varsigma}u_{\xi\varsigma} + 3\varphi_u u_{\varsigma}u_{\varsigma\varsigma} + \tau_u u_{\xi}u_{\varsigma\varsigma} + 2\tau_u u_{\varsigma}u_{\xi\varsigma}
\end{aligned}$$

The equation is then expanded and categorized by the derivatives of u.

u_{ξ}: $\quad 2\phi_{tu} - \tau_{tt} - a\tau_t + \tau_{xx} = 0$

u_{ς}: $\quad a\xi_t - \xi_{tt} - 2\phi_{xu} + \xi_{xx} = 0$

$u_{\varsigma}u_{\xi}$: $\quad -2\xi_{tu} + 2\tau_{xu} = 0$

$u_{\xi}{}^{2}$: $\quad 3a\phi_t + \phi_{uu} - 2\tau_{tu} - 2a\tau_u = 0$

${u_\xi}^3$:	$2a\phi_u - a\tau_t - \tau_{uu} = 0$
$u_\varsigma {u_\xi}^2$:	$-3a\xi_t - \xi_{uu} = 0$
$u_\varsigma {u_\xi}^3$:	$-2a\xi_u = 0$
${u_\xi}^4$:	0
$u_{\varsigma\varsigma}$:	$-2\tau_t + 2\xi_x = 0$
$u_{\xi\varsigma}$:	$-2\xi_t + 2\tau_x = 0$
$u_\xi u_{\varsigma\varsigma}$:	$-2\tau_u = 0$
$u_\varsigma u_{\varsigma\varsigma}$:	$2\xi_u = 0$
$u_\xi u_{\xi\varsigma}$:	$-2\xi_u = 0$
${u_\varsigma}^2$:	$-\phi_{uu} + 2\xi_{xu} = 0$
${u_\varsigma}^3$:	$\xi_{uu} = 0$
${u_\varsigma}^2 u_\xi$:	$\tau_{uu} = 0$
$u_\varsigma u_{\xi\varsigma}$:	$2\tau_u = 0$
Remainder:	$-a\phi_\xi + \phi_{\xi\xi} - \phi_{\varsigma\varsigma} = 0$

After simplified all the equation above, obtain

$$\varphi(\varsigma, \xi, u) = c_5,$$

$$\tau(\varsigma, \xi, u) = c_2,$$

$$\phi(\varsigma, \xi, u) = c_3\varsigma + c_4,$$

Where, c_2, c_3, c_4, and c_5 are arbitrary constants. Hence, the symmetry of equation (1) are

$$X_1 = \frac{\partial}{\partial u}, \qquad X_2 = \frac{\partial}{\partial \varsigma}, \qquad X_3 = \frac{\partial}{\partial \xi}, \qquad X_4 = \varsigma\frac{\partial}{\partial u}.$$

III. SYMMETRY REDUCTION

By taking different combination of symmetry, three reduction were conducted in order to obtain exact solutions of equation (1).

A. Reduction By X_2 and X_3

By taking combination of X_2 and X_3,

$$X = \frac{\partial}{\partial \varsigma} + \frac{\partial}{\partial \xi}.$$

To find the invariants, the corresponding characteristic equation is

$$\frac{d\varsigma}{1} = \frac{d\xi}{1} = \frac{du}{0}.$$

Hence, similarity equation obtained as

$$\beta = \varsigma - \xi,$$

$$\alpha = u.$$

Derivatives as below

$$u_\xi = -\alpha_\beta,$$

$$u_{\xi\xi} = \alpha_{\beta\beta},$$

$$u_\varsigma = \alpha_\beta,$$

$$u_{\varsigma\varsigma} = \alpha_{\beta\beta}.$$

Substitute all the derivatives of u into equation (1) to obtain a first order ordinary differential equation after the second order derivatives have been cancelled off.

$$\alpha_{\beta\beta} - \alpha_{\beta\beta} - \varepsilon(-\alpha_\beta) + \varepsilon(-\alpha_\beta)^3 = 0$$

$$\varepsilon(\alpha_\beta) - \varepsilon(\alpha_\beta)^3 = 0$$

Hence the equation have the reduced form of

$$\alpha_\beta = 0, \qquad \alpha_\beta = 1, \qquad \alpha_\beta = -1\,,$$

The first reduced equation give solution of

$$\alpha = A$$

where A is abirtary constant. Hence the first exact solution of equation (1) is

$$u = A. \qquad (4)$$

The second reduced equation give solution

$$\alpha_\beta = 1$$

$$\alpha = \beta + B$$

where B is abitary constant. After transform, solution for equation (1) is

$$u = \varsigma - \xi + B. \qquad (5)$$

The third reduced equation give solution

$$\alpha_\beta = -1$$

$$\alpha = -\beta + C$$

where C is abitary constant. After transform, solution for equation (1) is

$$u = \xi - \varsigma + C. \qquad (6)$$

B. Reduction By X_1, X_2 and X_3

By taking combination of X_1, X_2 and X_3,

$$X = \frac{\partial}{\partial u} + \frac{\partial}{\partial \varsigma} + \frac{\partial}{\partial \xi}.$$

To find the invariants, the corresponding characteristic equation is

$$\frac{d\varsigma}{1} = \frac{d\xi}{1} = \frac{du}{1}.$$

Hence, similarity equation obtained as

$$\alpha = u - \varsigma,$$

$$\beta = \varsigma - \xi.$$

Derivatives as below

$$u_\xi = -\alpha_\beta,$$

$$u_{\xi\xi} = \alpha_{\beta\beta},$$

$$u_\varsigma = 1 + \alpha_\beta,$$

$$u_{\varsigma\varsigma} = \alpha_{\beta\beta}.$$

Substitute all the derivatives of u into equation (1) to obtain a first order ordinary differential equation after the second order derivatives have been cancelled off.

$$\alpha_{\beta\beta} - \alpha_{\beta\beta} - \varepsilon(-\alpha_\beta) + \varepsilon(-\alpha_\beta)^3 = 0$$

$$\varepsilon(\alpha_\beta) - \varepsilon(\alpha_\beta)^3 = 0$$

Hence the equation have the reduced form of

$$\alpha_\beta = 0, \qquad \alpha_\beta = 1, \qquad \alpha_\beta = -1\,,$$

The first reduced equation give solution of

$$\alpha = D$$

where D is abirtary constant. Hence the first exact solution of equation (1) is

$$u = \varsigma + D. \qquad (7)$$

The second reduced equation give solution

$$\alpha_\beta = 1$$

$$\alpha = \beta + E$$

where E is abitary constant. After transform, solution for equation (1) is

$$u = \varsigma - \xi + E. \qquad (8)$$

This solution is same as the second solution for first combination.

The third reduced equation give solution

$$\alpha_\beta = -1$$

$$\alpha = -\beta + F$$

where F is abitary constant. After transform, solution for equation (1) is

$$u = \xi + F. \qquad (9)$$

C. Reduction By X_2 and X_4

By taking combination of X_2 and X_4,

$$X = \frac{\partial}{\partial\varsigma} + \varsigma\frac{\partial}{\partial u}.$$

To find the invariants, the corresponding characteristic equation is

$$\frac{d\varsigma}{1} = \frac{d\xi}{0} = \frac{du}{\varsigma}.$$

Hence, similarity equation obtained as

$$\alpha = \frac{\varsigma^2}{2} - u,$$

$$\beta = \xi.$$

Derivatives as below

$$u_\xi = -\alpha_\beta,$$

$$u_{\xi\xi} = -\alpha_{\beta\beta},$$

$$u_\varsigma = \varsigma,$$

$$u_{\varsigma\varsigma} = 1.$$

Substitute all the derivatives of u into equation (1) to obtain a first order ordinary differential equation after the second order derivatives have been cancelled off.

$$-\alpha_{\beta\beta} - 1 - \varepsilon(-\alpha_\beta) + \varepsilon(-\alpha_\beta)^3 = 0$$

$$-\alpha_{\beta\beta} - 1 + \varepsilon(\alpha_\beta) - \varepsilon(\alpha_\beta)^3 = 0$$

This is a second order ordinary differeantial equation (ODE), to reduce this equation into first order ODE, Lie symmetry reduction been used again, take $M = \alpha$ and $N = \alpha_\beta$, compute the following:

$$\frac{dN}{dM} = \frac{\left(\frac{dN}{d\alpha}\right)}{\left(\frac{dM}{d\alpha}\right)} = \frac{\left(\frac{dN}{d\alpha}\right)}{1} = \frac{dN}{d\alpha} = \frac{\left(\frac{dN}{d\beta}\right)}{\left(\frac{d\alpha}{d\beta}\right)} = \frac{\alpha_{\beta\beta}}{N}$$

$$N\frac{dN}{dM} = \alpha_{\beta\beta}\ .$$

Substitute these variables into the second order ODE and get

$$-N\frac{dN}{dM} - 1 + \varepsilon N - \varepsilon N^3 = 0,$$

then rearrange into the equation as follow:

$$\frac{dN}{dM} = \frac{\varepsilon N^3 - \varepsilon N + 1}{-N},$$

where gives the solution of

$$N(M) = \pm\sqrt{-2\left(\int \varepsilon N(M)^3\, dM\right) + 2\left(\int \varepsilon N(M)\, dM\right) - 2M + G}$$

where G is abirtrary constant.

Transform back, following solution of equation (1) are obtained

$$\alpha_\beta = \pm\sqrt{-2\left(\int \varepsilon\alpha_\beta^3\, d\alpha\right) + 2\left(\int \varepsilon\alpha_\beta\, d\alpha\right) - 2\alpha + G}$$

$$\alpha = \pm\int\sqrt{-2\left(\int \varepsilon\alpha_\beta^3\, d\alpha\right) + 2\left(\int \varepsilon\alpha_\beta\, d\alpha\right) - 2\alpha + G}\, d\beta$$

$$\frac{\varsigma^2}{2} - u = \pm\int\sqrt{2\left(\int \varepsilon u_\xi^3\, d\left(\frac{\varsigma^2}{2} - u\right)\right) - 2\left(\int \varepsilon u_\xi\, d\left(\frac{\varsigma^2}{2} - u\right)\right) - 2(\frac{\varsigma^2}{2} - u) + G}\, d\xi \tag{10}$$

IV. Discussion and conclusion

By taking three associations between the symmetries are investigated which are the combinations of symmetry X_2 and X_3, symmetry X_1, X_2 and X_3 as well as symmetry X_2 and X_4 respectively. The combinations of symmetries are then solved by transformation and reduction to give solutions as below:

$$u = A,\ u = \varsigma - \xi + B,\ u = \xi - \varsigma + C,\ u = \varsigma + A,$$

$$u = \varsigma - \xi + B,\ u = \xi + C,$$

$$\frac{\varsigma^2}{2} - u = \pm \int \sqrt{2\left(\int \varepsilon u_\xi^3\, d\left(\frac{\varsigma^2}{2} - u\right)\right) - 2\left(\int \varepsilon u_\xi\, d\left(\frac{\varsigma^2}{2} - u\right)\right) - 2(\frac{\varsigma^2}{2} - u) + G}\, d\xi$$

These process had successfully reduce an second order PDE into ODE and hence obtained the exact solution. However, comparison between the results yielded by this project and other researches could not be done to check for its precision as the results are in different forms compare to research done before.

Acknowledgment

Author wish to ackowledg helping of UiTM in providing fund to prepare this paper. Also, author wish to thanks for the help from all the co-author and colleuges for providing advice in preparing the paper.

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