

# Formulation Method of Gain Calculation at Marginal Stability of a Linear Invariant Control Systems

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**Abstract:** The stability analysis of a linear invariant control system is based mainly on its characteristic equation. There are various methods to examine, to do analysis, and to design a control system. Those methods become less effective and more complicated for use when those methods are used for a high order of an open loop transfer function that has several poles and several zeros. In this research work a new method was developed to find the gains at marginal stability and the intersection points with the imaginary axis of a single-input and single-output of a linear invariant control system by using two new formulas. First formula is used to construct a new polynomial where its roots are the intersection points with the imaginary axis of the  $s$ -plane, and a second formula is used to calculate the gains at the marginal stability of the system. The coefficients of the characteristic equation's polynomial of the control system are substituted in the first formula to obtain a new polynomial. The roots of the obtained polynomial are substituted in the second formula to obtain the gains at marginal stability. In this research work the derivation of the polynomial's construction formula, its mathematical proof, and the derivation of the gains formula at marginal stability are presented. The proposed Formulization method is compared with another three common methods in the solution of three examples. The used methods are the proposed method, Routh-Hurwitz criterion, Root Locus technique, and the complex variable  $s$  on the imaginary axis. The chosen examples are three control systems where their transfer functions are different in order and in complexity, going from low to high. The comparison shows that the Formulization method is accurate and needs less mathematical operations by the user. It is applicable for any order of a single-input and single-output of invariant control systems. It is an effective method especially for a higher order and for more complicated transfer functions of the control systems.

**Keywords:** Gains at marginal stability; Gains calculation at the intersection points; Intersection points with the imaginary axis; Marginal stability; Polynomial of intersection points.

## 1. INTRODUCTION

The characteristic equation of the closed loop control system is obtained by equating the denominator of its transfer function to zero. The denominator's roots of the open loop transfer function are called the poles and the roots of the numerator are called the zeros. One of the most important properties of a control system is its stability. In theory, a time invariant control system is stable if its output remains bounded when excited by a bounded input, Bounded Input Bounded Output (BIBO). Unstable system is a system in which its output does not return to steady state for a bounded change in its input. The system's state between the two previous states is the border of stability, and it is called a state of marginal stability of the system [1, 2]. A conditionally stable system if it is stable for a certain range of gain's values.

The absolute stability of a system is referred to a stable system or unstable system, while the system relative stability is referred to how far away be the system's stability from its border of stability. System stability can be classified based on its type of stability such as absolute stability of the system, conditional stability of the system, and marginal stability of the system. Absolutely stable system if all its poles are in the left half of the  $s$ -plane [3]. While a marginally stable system if its output to a bounded input may have a bounded offset, or a sustained oscillations of constant amplitude and constant frequency. Theoretically the system becomes a marginally stable system if it has non-repeated imaginary poles with zero real parts.

It is known that at the state of marginal stability any perturbed input of the control system moves the system to an unstable state. Therefore, this state of stability is a risky state. Consequently, the design of control systems is to keep the system away from this state. This state is very important to know because it is the border of system stability. The location of the poles of the system's transfer function on the complex plane,  $s$ -plane, determines the system stability. To have a stable control system requires that all roots of the closed loop characteristic equation are located on the left side of the  $s$ -plane, or mathematically, the real part of every root must be negative. If there are roots with zero real parts and non-repeated parts, they are on the

imaginary axis of the s-plane, and the system is marginally stable. Therefore, moving the system transfer function poles to the left of the imaginary axis of the s-plane increases the system relative stability [4]. To avoid loss of control, and may be a damage to equipment, the system internal signal must be bounded and give a bounded output. This means that the system is BIBO stable. Stability of linear feedback systems can be determined by the real part of its closed-loop transfer function's poles.

For practical analysis and design of a control system it is more useful to know the relative stability, which means how much the loop gain can be changed before the system becomes unstable, gain margin. One way to assess the relative stability of a system is to use root locus technique to find the range of the gain values for a stable system [5, 6]. Root locus technique shows the effect of the system gain value on the system stability and the transient response of the system. The gain range is determined by knowing the intersection points of the root loci with the imaginary axis on the s-plane. At the intersection points, the control system stability may change state from stable to unstable or versa. One method for finding the intersection points and the corresponding gains' values is Routh-Hurwitz criterion [7]. Routh-Hurwitz stability criterion is an analytical technique that gives the number of roots with positive real part, where poles with a positive real part are the unstable roots, [8, 9]. The desired performance of a control system is achieved by tuning its controller's gains. A basic tuning method and its modifications are based on the ultimate gain values at the intersection points with the imaginary axis, where the system is marginally stable. As a result, there is a necessity to obtain the limit value of the gain.

The focus in this research work was the development of a new method for finding the imaginary roots of the characteristic equation and the corresponding marginal gains by using derived two formulas. One formula is for constructing a new polynomial where its roots are the intersection points with the imaginary axis of the s-plane, and a second formula is used to calculate the corresponding gains for a single-input single-output of a linear invariant control system. The expected outcome is having a new method to calculate the critical gain and imaginary axis crossing, which is easier, systematic process, and more flexible for higher order systems.

## 2. THE DERIVATION OF THE CONDITIONS FOR ZERO RESIDUE

Based on the Routh-Hurwitz criterion, the row which proceeds a row of zeros in the Routh array gives a polynomial called the auxiliary equation. The auxiliary equation is a factor of the characteristic equation of the system. In this case there are roots of the characteristic equation equally spaced from the origin of the s-plane. The equally spaced roots may be on the imaginary axis. If the roots of the auxiliary equation are an imaginary number with zero real part, then the root locus graph crosses the imaginary axis. The simple form of the auxiliary equation for imaginary roots is

$$(s + j\omega)(s - j\omega) = s^2 + \omega^2 \quad (1)$$

If the polynomial, Equation (1), is a factor of the characteristic equation, the residue of the division of the characteristic equation by this factor is zero. In this research work the general form of characteristic equations for various degrees are divided on the polynomial, Equation (1), to get the division residues. Setting the expression of the residue to zero gives two conditions for being Equation (1) a factor of the characteristic equation. Those conditions are two algebraic equations which are based on the coefficients of the denominator's polynomial and numerator polynomial of the system open loop transfer function.

The characteristic equation of a closed loop of a single input-single output control system is

$$1 + KG(s)H(s) = 0. \quad (2)$$

where  $G(s)$  is the forward transfer function and  $H(s)$  is the feedback of the control system, Figure 1. In this case the parameter of interest is the static gain  $K$ . The rational form of a single input-single output system's transfer function,  $G(s)H(s)$  is

$$G(s)H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}. \quad (3)$$

Substitution of Equation (3) into Equation (2) gives an explicit form of the characteristic equation as a function of the complex variable  $s$ . The coefficients of the characteristic equation's polynomial are the coefficients of the open loop transfer function's polynomials such as

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0) + K(b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0) = 0. \quad (4)$$

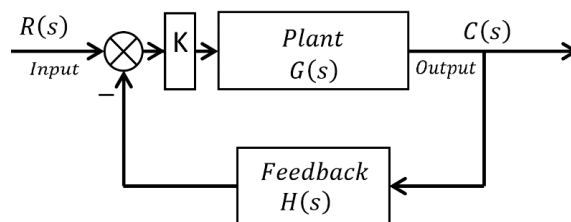


Figure 1. Block diagram of single input single output of control system

The general form of Equation (4) is

$$c_n s^n + c_{n-1} s^{n-1} + \dots + c_1 s + c_o = 0. \tag{5}$$

where

$$c_i = a_i + K b_i. \tag{6}$$

As an example of the derivation of the two conditions for being Equation (1) a factor of the characteristic equation, a third-degree characteristic equation is considered. The general form of a third-degree characteristic equation is

$$c_3 s^3 + c_2 s^2 + c_1 s + c_o = 0. \tag{7}$$

The remainder from a long division of Equation (7) as the dividend, on Equation (1) as the divisor, is

$$(c_1 - c_3 \omega^2) s + (c_o - c_2 \omega^2). \tag{8}$$

To satisfy the condition that Equation (1) is a factor for a third-degree characteristic polynomial, the two parts of the division's remainder are set to zero. By doing that we obtain two conditions, and they are

$$\begin{aligned} (c_1 - c_3 \omega^2) &= 0, \\ (c_o - c_2 \omega^2) &= 0. \end{aligned} \tag{9}$$

The solution for  $\omega$  gives the values of the intersection points on the imaginary axis. Repetition of the division on Equation (1) of a higher degree of characteristic equations gives a similar result. The results for the quotients of the division are shown in Table 1, and the remainders are shown as two conditions in Table 2.

The mathematical proof which shows that the general form of degree  $n$  of the results in Table 1 and Table 2 are correct is shown in Appendix A.

Table 1. The quotients of the long division for polynomials of degree 2,3, ..., n.

Degree n	Divisor	$Q_n$ : Quotient
2	$s^2 + \omega^2$	$c_2$
3	$s^2 + \omega^2$	$c_3 s + c_2$
4	$s^2 + \omega^2$	$c_4 s^2 + c_3 s + c_2$ $-(c_4) \omega^2$
5	$s^2 + \omega^2$	$c_5 s^3 + c_4 s^2 + c_3 s + c_2$ $-(c_5 s + c_4) \omega^2$
6	$s^2 + \omega^2$	$c_6 s^4 + c_5 s^3 + c_4 s^2 + c_3 s + c_2$ $-(c_6 s^2 + c_5 s + c_4) \omega^2$ $+(c_6) \omega^4$
7	$s^2 + \omega^2$	$c_7 s^5 + c_6 s^4 + c_5 s^3 + c_4 s^2 + c_3 s + c_2$ $-(c_7 s^3 + c_6 s^2 + c_5 s + c_4) \omega^2$ $+(c_7 s + c_6) \omega^4$
8	$s^2 + \omega^2$	$c_8 s^6 + c_7 s^5 + c_6 s^4 + c_5 s^3 + c_4 s^2 + c_3 s + c_2$ $-(c_8 s^4 + c_7 s^3 + c_6 s^2 + c_5 s + c_4) \omega^2$ $+(c_8 s^2 + c_7 s + c_6) \omega^4$ $-(c_8 s^0) \omega^6$
⋮		⋮
n	$s^2 + \omega^2$	$Q_n = Q_{n-1} + C_n \sum_{m=2,4,\dots}^n (-1)^{\frac{m}{2}+1} s^{n-m} \omega^{m-2}$ Or $Q_n = \sum_{i=2,3,\dots}^n C_i \sum_{m=2,4,\dots}^i (-1)^{\frac{m}{2}+1} s^{i-m} \omega^{m-2}$

Table 2. The two conditions for being  $(s^2 + \omega^2)$  a factor for a polynomial of degree 2,3, ..., n.

Degree n	R <sub>1</sub> : First Condition	R <sub>2</sub> : Second Condition
2	$c_0 - c_2\omega^2 = 0$	$c_1 = 0$
3	$c_0 - c_2\omega^2 = 0$	$c_1 - c_3\omega^2 = 0$
4	$c_0 - c_2\omega^2 + c_4\omega^4 = 0$	$c_1 - c_3\omega^2 = 0$
5	$c_0 - c_2\omega^2 + c_4\omega^4 = 0$	$c_1 - c_3\omega^2 + c_5\omega^4 = 0$
6	$c_0 - c_2\omega^2 + c_4\omega^4 - c_6\omega^6 = 0$	$c_1 - c_3\omega^2 + c_5\omega^4 = 0$
7	$c_0 - c_2\omega^2 + c_4\omega^4 - c_6\omega^6 = 0$	$c_1 - c_3\omega^2 + c_5\omega^4 - c_7\omega^6 = 0$
⋮	⋮	⋮
n	$R_1 = \sum_{m=0,2,4}^n (-1)^{\frac{m}{2}} c_m \omega^m = 0$	$R_2 = \sum_{m=1,3,5}^n (-1)^{\frac{m+3}{2}} c_m \omega^{m-1} = 0$

Table 3. The gains formula for degrees 2,3, ..., n.

n	K first formula	K second formula
2	$K = -\frac{a_0 - a_2\omega^2}{b_0 - b_2\omega^2}$	$K = -\frac{a_1}{b_1}$
3	$K = -\frac{a_0 - a_2\omega^2}{b_0 - b_2\omega^2}$	$K = -\frac{a_1 - a_3\omega^2}{b_1 - b_3\omega^2}$
4	$K = -\frac{a_0 - a_2\omega^2 + a_4\omega^4}{b_0 - b_2\omega^2 + b_4\omega^4}$	$K = -\frac{a_1 - a_3\omega^2}{b_1 - b_3\omega^2}$
5	$K = -\frac{a_0 - a_2\omega^2 + a_4\omega^4}{b_0 - b_2\omega^2 + b_4\omega^4}$	$K = -\frac{a_1 - a_3\omega^2 + a_5\omega^4}{b_1 - b_3\omega^2 + b_5\omega^4}$
6	$K = -\frac{a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6}{b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6}$	$K = -\frac{a_1 - a_3\omega^2 + a_5\omega^4}{b_1 - b_3\omega^2 + b_5\omega^4}$
7	$K = -\frac{a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6}{b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6}$	$K = -\frac{a_1 - a_3\omega^2 + a_5\omega^4 - a_7\omega^6}{b_1 - b_3\omega^2 + b_5\omega^4 - b_7\omega^6}$
⋮	⋮	⋮
n	$K = -\frac{\sum_{m=0,2,4}^n (-1)^{\frac{m}{2}} a_m \omega^m}{\sum_{m=0,2,4}^n (-1)^{\frac{m}{2}} b_m \omega^m}$	$K = -\frac{\sum_{m=1,3,5}^n (-1)^{\frac{m+3}{2}} a_m \omega^{m-1}}{\sum_{m=1,3,5}^n (-1)^{\frac{m+3}{2}} b_m \omega^{m-1}}$

### 3. DERIVATION OF THE PROPOSED METHOD FORMULAS

#### 3.1 Derivation of the Marginal Gain's Formula

For an example a third order system is considered. The substitution of Equation (6) into Equation (8) gives two algebraic equations with two unknowns such as

$$a_1 + Kb_1 - (a_3 + Kb_3)\omega^2 = 0. \tag{10}$$

$$a_0 + Kb_0 - (a_2 + Kb_2)\omega^2 = 0. \tag{11}$$

Solve both equations for K to get

$$K = -\frac{a_0 - a_2\omega^2}{b_0 - b_2\omega^2}. \tag{12}$$

$$K = -\frac{a_1 - a_3\omega^2}{b_1 - b_3\omega^2}. \tag{13}$$

We obtained two formulas for the gain K and the used one is the one with a nonzero dominator. Repeating this procedure for higher degree of characteristic equations gives similar results that are shown in Table 3.

#### 3.2 Derivation of the Intersection Points' Formula

Here is a derivation of a formula that is used to construct a polynomial where its roots give the intersection points with the imaginary axis of the s-plane. The derivation is accomplished by equating the two expressions of the two formulas for each degree in Table 3. The obtained polynomial is a function of independent variable omega as shown in Table 4. Those points (omega values) are points on the imaginary axis of the s-plane where the system will be at a marginal stability. Then omega values are substituted in the corresponding formula that is given in Table 3 to obtain the marginal gains.

Table 4. Formula of the polynomial of the intersection points for a system of degrees 2,3, ..., n.

n	Polynomial Formula
2	$(a_0 - a_2\omega^2)(b_1) - (b_0 - b_2\omega^2)(a_1) = 0$
3	$(a_0 - a_2\omega^2)(b_1 - b_3\omega^2) - (b_0 - b_2\omega^2)(a_1 - a_3\omega^2) = 0$
4	$(a_0 - a_2\omega^2 + a_4\omega^4)(b_1 - b_3\omega^2) - (b_0 - b_2\omega^2 + b_4\omega^4)(a_1 - a_3\omega^2) = 0$
5	$(a_0 - a_2\omega^2 + a_4\omega^4)(b_1 - b_3\omega^2 + b_5\omega^4) - (b_0 - b_2\omega^2 + b_4\omega^4)(a_1 - a_3\omega^2 + a_5\omega^4) = 0$
6	$(a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6)(b_1 - b_3\omega^2 + b_5\omega^4) - (b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6)(a_1 - a_3\omega^2 + a_5\omega^4) = 0$
7	$(a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6)(b_1 - b_3\omega^2 + b_5\omega^4 - b_7\omega^6) - (b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6)(a_1 - a_3\omega^2 + a_5\omega^4 - a_7\omega^6) = 0$
⋮	⋮
n	$(\sum_{m=0,2,4}^n (-1)^{\frac{m}{2}} a_m \omega^m) (\sum_{m=1,3,5}^n (-1)^{\frac{m+3}{2}} b_m \omega^{m-1}) - (\sum_{m=0,2,4}^n (-1)^{\frac{m}{2}} b_m \omega^m) (\sum_{m=1,3,5}^n (-1)^{\frac{m+3}{2}} a_m \omega^{m-1}) = 0$

#### 4. COMPARISON OF THE PROPOSED METHOD WITH COMMON METHODS

The proposed method is compared with the other three common methods. The comparison is done by showing the finding of the intersection points with the imaginary axis and the marginal gains in the following three examples. The order of methods is: 1) The proposed Marginal Gain-Polynomial of Intersection Points' formulas, 2) Routh-Hurwitz criterion, 3) Root Locus plot using MATLAB, and 4)  $s = j\omega$  where the  $s$  variable in the characteristic equation of the control system has zero real part.

##### 4.1 Example 1

In this example a third order system is considered. This system has two crossing points with the imaginary axis of the  $s$ -plane. The transfer function of the system is

$$G(s)H(s) = K \frac{1}{(s + 1)(s + 3)(s + 5)}. \tag{14}$$

The polynomials of the denominator and the numerator are

$$D(s) = s^3 + 9s^2 + 23s + 15, \quad N(s) = 1. \tag{15}$$

##### 4.1.1 Using the Proposed Formulas

From Table 4 the formula of the polynomial of intersection points for  $n = 3$  is

$$(a_0 - a_2\omega^2)(b_1 - b_3\omega^2) - (a_1 - a_3\omega^2)(b_0 - b_2\omega^2) = 0. \tag{16}$$

The substitution of the polynomials' coefficients of the transfer function into Equation (16) gives a polynomial. The roots of this polynomial give the intersection points with the imaginary axis. In this example the polynomial and the intersection points are

$$(15 - 9\omega^2)(0) - (23 - \omega^2)(1) = 0 \rightarrow \omega = \pm\sqrt{23} \rightarrow \text{intersection points } s = \pm j\sqrt{23}. \tag{17}$$

Substitution of omega value in the marginal gain formula which is given in Table 3 gives:

$$K = -\frac{a_0 - a_2\omega^2}{b_0 - b_2\omega^2} = -\frac{15 - 9(23)}{1} = 192. \tag{18}$$

##### 4.1.2 Using Routh-Hurwitz Criterion

The characteristic equation is:

$$s^3 + 9s^2 + 23s + 15 + K = 0. \tag{19}$$

The Routh array is shown in Figure 2. Based on this criterion the gain for marginal stability is  $K=192$ , and the auxiliary equation proceeds the row of zeros is

$$9s^2 + 15 + k = 9s^2 + 15 + 192 = 9s^2 + 207 = 0. \tag{20}$$

$S^3$	1	23
$S^2$	9	$15+K$
$S^1$	$K-192$	0
$S^0$	$15+K$	0

Figure 2. Routh Table of Example 1

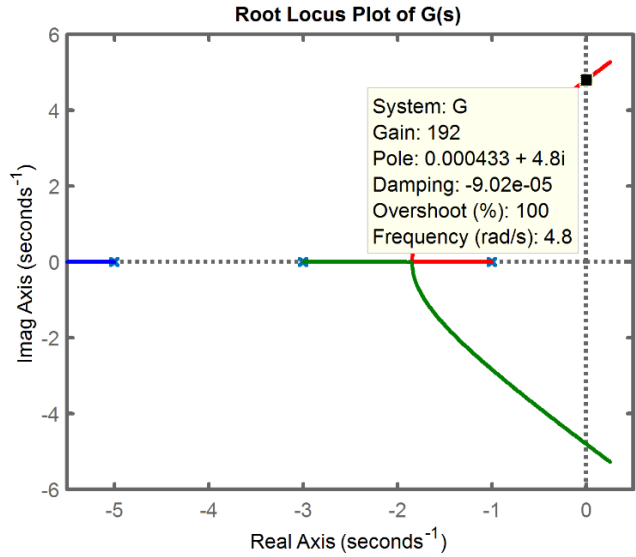


Figure 3. Root Locus of Example 1

Then the crossing points with the imaginary axis are

$$s^2 + \frac{207}{9} = 0 \rightarrow s_{1,2} = \sqrt{-23} = \pm j\sqrt{23}. \quad (21)$$

#### 4.1.3 Using Root Locus Technique

Using MATLAB to plot the root locus. The result is shown in Figure 3.

#### 4.1.4 The Complex Variable $s$ is an Imaginary Variable

The value of the complex variable  $s$  on the imaginary axis has zero real part,  $s = j\omega$ . Substitution of  $s$  in the characteristic equation, Equation (19), gives

$$s^3 + 9s^2 + 23s + 15 + K = -j\omega^3 - 9\omega^2 + j23\omega + 15 + K = 0. \quad (22)$$

To satisfy Equation (22) to have just an imaginary number, implies that the real part of the equation is zero. Consequently, the imaginary part of the equation is also zero. The result is that we obtain two equations. The two equations in this example are

$$-9\omega^2 + 15 + K = 0 \quad \text{and} \quad -j\omega^3 + j23\omega = 0. \quad (23)$$

Solve the second part for omega and then substitute omega value in the first part to obtain the gain  $K$  as

$$\omega = \pm\sqrt{23} \quad \text{and} \quad K = 9\omega^2 - 15 = 207 - 15 = 192. \quad (24)$$

## 4.2 Example 2

In this example a fifth order system is considered, and to increase its complexity its transfer function has one zero. This system has two intersection points with the imaginary axis of the  $s$ -plane. The system's transfer function is

$$G(s)H(s) = K \frac{s + 3}{s(s + 5)(s + 6)(s^2 + 2s + 2)}. \quad (25)$$

The polynomials of the denominator and the numerator are

$$D(s) = s^5 + 13s^4 + 54s^3 + 82s^2 + 60s, \quad N(s) = s + 3. \quad (26)$$

#### 4.2.1 Using the Proposed Formulas

Using the polynomial formula given in Table 4 for  $n = 5$  is

$$(a_0 - a_2\omega^2 + a_4\omega^4)(b_1 - b_3\omega^2 + b_5\omega^4), \\ -(b_0 - b_2\omega^2 + b_4\omega^4)(a_1 - a_3\omega^2 + a_5\omega^4) = 0. \quad (27)$$

Substitution of the polynomials' coefficients of the transfer function into Equation (27) gives

$$(-82\omega^2 + 13\omega^4)(1) - (3)(60 - 54\omega^2 + \omega^4) = 0. \tag{28}$$

After simplification and then solving for omega to get

$$\omega^4 + 8\omega^2 - 18 = 0 \rightarrow \text{real } \omega = \pm 1.3531 \rightarrow s = \pm j1.3531. \tag{29}$$

Then substitution of omega in the gain K formula for n = 5 which is given in Table 3 gives the marginal gain as

$$K = -\frac{a_0 - a_2\omega^2 + a_4\omega^4}{b_0 - b_2\omega^2 + b_4\omega^4} = -\frac{0 - 82\omega^2 + 13\omega^4}{3 - 0 - 0} = \frac{(82 - 13 \times 1.831)1.831}{3} = 35.519. \tag{30}$$

4.2.2 Using Routh-Hurwitz Criterion

The characteristic equation of this example is

$$s^5 + 13s^4 + 54s^3 + 82s^2 + (60 + K)s + 3K = 0. \tag{31}$$

To have a row of zeros the expression in the fifth row that is shown in Figure 4 should be zero, as a result we get

$$24,420 - 652K - K^2 = 0. \tag{32}$$

Solve to get the gain K of marginal stability as K=35.5190. Then construction of the auxiliary equation gives

$$\frac{40,700 - 130K}{620}s^2 + 3k = 0 \rightarrow \frac{40,700 - 130(35.519)}{620}s^2 + 3(35.519) = 0. \tag{33}$$

Then solution of the auxiliary equation gives the intersection points as:  $s = \pm j1.3531$ .

4.2.3 Using Root Locus Technique

Using MATLAB to plot the root locus of this example. The result is shown on the graph of Figure 5.

4.2.4 The Complex Variable s is an Imaginary Variable

The characteristic equation on the imaginary axis is obtained by substituting  $s = j\omega$  into the characteristic equation, Equation (31), as

$$j\omega^5 + 13\omega^4 - j54\omega^3 - 82\omega^2 + j(60 + K)\omega + 3K = 0. \tag{34}$$

S <sup>5</sup>	1	54	60+K
S <sup>4</sup>	13	82	3K
S <sup>3</sup>	$\frac{620}{13}$	$\frac{780 + 10K}{13}$	0
S <sup>2</sup>	$\frac{40,700 - 130K}{620}$	3K	0
S <sup>1</sup>	$\frac{24,420 - 652K - K^2}{407 - 1.3K}$	0	0
S <sup>0</sup>	3K		

Figure 4. Routh array of Example 2

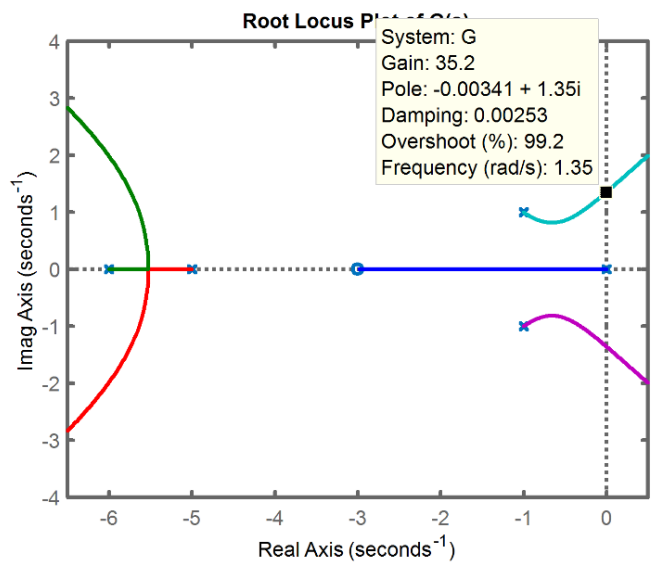


Figure 5. Root Locus of Example 2

To satisfy this equation implies that its two parts are zeros, so the result there are two equations such as

$$13\omega^4 - 82\omega^2 + 3K = 0, \text{ and } j\omega^5 - j54\omega^3 + j(60 + K)\omega = 0. \quad (35)$$

Simplification of the second part and rewrite Equation (35) to have the two parts such as

$$13\omega^4 - 82\omega^2 + 3K = 0, \text{ and } \omega^4 - 54\omega^2 + 60 + K = 0. \quad (36)$$

Elimination of  $K$  from the two parts to get a polynomial of omega. Solving the obtained polynomial for real values of omega and then the intersection points, and then the substitution of omega in one of the two parts gives the marginal gain at the intersection points such as

$$\omega = \pm 1.3531 \rightarrow \text{Intersection points: } s = \pm j1.3531, \text{ and the gain is } K = 35.519. \quad (37)$$

### 4.3 Example 3

In this example the system's order is seven. It has four zeros to increase its complexity significantly. This system plot has eight intersection points with the imaginary axis in the  $s$ -plane. The transfer function of this system in example is

$$G(s)H(s) = K \frac{(s^2 + s + 0.25)(s^2 + 2.5s + 65.56)}{s^3(s^2 + 2.4s + 160)(s^2 + 6s + 18)}. \quad (38)$$

The polynomials of the denominator and the numerator of the transfer function are:

$$\begin{aligned} D(s) &= s^7 + 30s^6 + 322s^5 + 1392s^4 + 2880s^3, \\ N(s) &= s^4 + 3.5s^3 + 68.31s^2 + 66.19s + 16.39. \end{aligned} \quad (39)$$

#### 4.3.1 Using the Proposed Formulas

From Table 4 the polynomial formula for  $n = 7$  is

$$\begin{aligned} (a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6)(b_1 - b_3\omega^2 + b_5\omega^4 - b_7\omega^6), \\ -(b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6)(a_1 - a_3\omega^2 + a_5\omega^4 - a_7\omega^6) = 0. \end{aligned} \quad (40)$$

Substitute the polynomials' coefficients of the transfer function into Equation (40) to obtain

$$\begin{aligned} (1392\omega^4 - 30\omega^6)(66.19 - 3.5\omega^2), \\ -(16.39 - 68.31\omega^2 + \omega^4)(-2880\omega^2 + 322\omega^4 - \omega^6) = 0. \end{aligned} \quad (41)$$

Simplify to get

$$\omega^2(\omega^8 - 285.31\omega^6 + 18,034.51\omega^4 - 109,873.9\omega^2 + 47,203.2) = 0. \quad (42)$$

The roots omegas of Equation (42) are

$$\omega = \pm 0, \pm 0.6818, \pm 2.5076, \pm 9.0713, \pm 14.0096. \quad (43)$$

Then the intersection points are

$$s_{1,2} = \pm j0.6818, s_{3,4} = \pm j2.5076, s_{5,6} = \pm j9.0713, s_{7,8} = \pm j14.0096. \quad (44)$$

The corresponding gains are calculated using the formula given in Table 3 for  $n = 7$ .

$$K = -\frac{a_0 - a_2\omega^2 + a_4\omega^4 - a_6\omega^6}{b_0 - b_2\omega^2 + b_4\omega^4 - b_6\omega^6} = -\frac{1392\omega^4 - 30\omega^6}{16.39 - 68.31\omega^2 + \omega^4} = \frac{\omega^4(30\omega^2 - 1392)}{\omega^4 - 68.31\omega^2 + 16.39}. \quad (45)$$

Substitute the omegas to get the corresponding marginal gains as

$$K_1 = 19.66, \quad K_2 = 127.35, \quad K_3 = 6,249, \quad K_4 = 6,891.8 \quad (46)$$

#### 4.3.2 Using Routh-Hurwitz Criterion

The characteristic equation of this example is:



S <sup>7</sup>	1	322	3.5K+2880	66.19K
S <sup>6</sup>	30	K+1392	68.31K	16.39K
S <sup>5</sup>	b <sub>1</sub>	b <sub>3</sub>	b <sub>5</sub>	0
S <sup>4</sup>	c <sub>1</sub>	c <sub>3</sub>	c <sub>5</sub>	0
S <sup>3</sup>	d <sub>1</sub>	d <sub>3</sub>	0	0
S <sup>2</sup>	e <sub>1</sub>	e <sub>3</sub>	0	0
S <sup>1</sup>	f <sub>1</sub>	0	0	0
S <sup>0</sup>	g <sub>1</sub>	0	0	0

Figure 6. Routh array of Example 3

$$s^7 + 30s^6 + 322s^5 + (K + 1392)s^4 + (3.5K + 2880)s^3 + 68.31Ks^2 + 66.19Ks + 16.39K = 0. \quad (47)$$

Based on the characteristic equation of this example, Equation (47) the Routh array is shown in Figure 6, where the elements of the array are

$$b_1 = -\frac{1}{30} \left| \begin{array}{cc} 1 & 322 \\ 30 & K + 1392 \end{array} \right| = \frac{8268 - K}{30},$$

$$b_3 = -\frac{1}{30} \left| \begin{array}{cc} 1 & 3.5K + 2880 \\ 30 & 68.31K \end{array} \right| = \frac{86,400 + 36.69K}{30},$$

$$b_5 = -\frac{1}{30} \left| \begin{array}{cc} 1 & 66.19K \\ 30 & 16.39K \end{array} \right| = 65.643K. \quad (48)$$

The consecutive expressions of the elements become more and more complicated and as an example

$$c_1 = \frac{-K^2 + 5,775.3K + 8,917,056}{K - 8,268}, \quad c_3 = \frac{K(68.31K - 505,707.78)}{K - 8,268}. \quad (49)$$

It is very hard to obtain the later elements' expressions. As a result, the Routh-Hurwitz criterion is not a useful method for a high degree of transfer function.

#### 4.3.3 Using Root Locus Technique

Using MATLAB to plot the root locus of Example 3 is shown Figures 7 and 8.

#### 4.3.4 The Complex Variable $s$ is an Imaginary Variable

Substitution into the characteristic equation, Equation (47),  $s = j\omega$  to get

$$(j\omega)^7 - 30(j\omega)^6 + 322(j\omega)^5 + (1392 + K)(j\omega)^4 + (2880 + 3.5K)(j\omega)^3 + 68.31K(j\omega)^2 + 66.19K(j\omega) + 16.39K = 0. \quad (50)$$

Or

$$j[-\omega^7 + 322\omega^5 - (2880 + 3.5K)\omega^3 + 66.19K\omega] + [-30\omega^6 + 1392\omega^4 + K(\omega^4 - 68.31\omega^2 + 16.39)] = 0. \quad (51)$$

If the first part is zero, then solve for  $K$  to have

$$K = \frac{-\omega^6 + 322\omega^4 - 2,880\omega^2}{3.5\omega^2 - 66.19}. \quad (52)$$

Substitution into the second term to get

$$-30\omega^6 + 1392\omega^4 + \frac{-\omega^6 + 322\omega^4 - 2,880\omega^2}{3.5\omega^2 - 66.19}(\omega^4 - 68.31\omega^2 + 16.39) = 0. \quad (53)$$

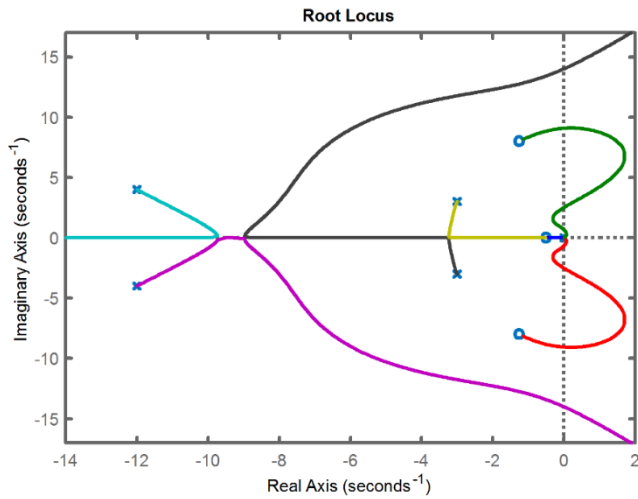


Figure 7. Root Locus plot of Example 3

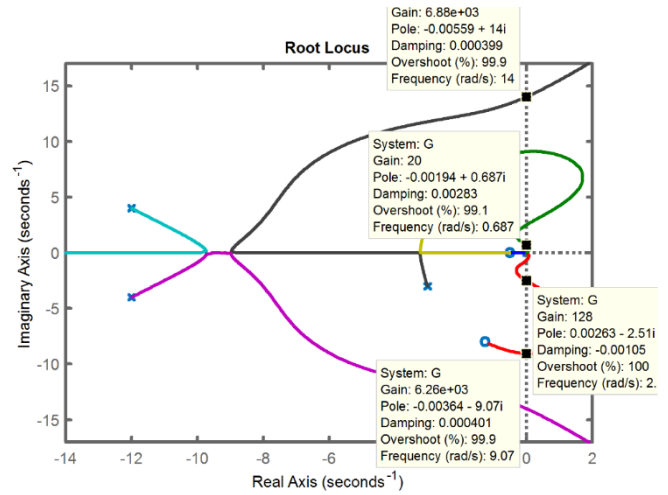


Figure 8. Root Locus of Example 3 showing the values at the intersection points with the imaginary axis

Or

$$(3.5\omega^2 - 66.19)(-30\omega^6 + 1392\omega^4) + , \\ +(-\omega^6 + 322\omega^4 - 2,880\omega^2)(\omega^4 - 68.31\omega^2 + 16.39) = 0. \quad (54)$$

The result of the multiplication and then of the simplification is

$$\omega^2(-\omega^8 + 285.31\omega^6 - 18,034.51\omega^4 + 109,873.9\omega^2 - 47,203.2) = 0. \quad (55)$$

The omega values and the intersection points are

$$\omega = \pm 0, \quad \pm 0.6818, \quad \pm 2.5076, \quad \pm 9.0713, \quad \pm 14.0096. \\ s_{1,2} = \pm j0.6816, \quad s_{3,4} = \pm j2.5084, \quad s_{5,6} = \pm j9.0691, \quad s_{7,8} = \pm j14.0105. \quad (56)$$

Substitution into Equation (52) to get the marginal gains as

$$K_1 = 19.66, \quad K_2 = 127.35, \quad K_3 = 6,249, \quad K_4 = 6,891.8 \quad (57)$$

## 5. DISCUSSION AND CONCLUSION

In principle the most commonly used methods by engineers and in classrooms for finding the intersection points with the imaginary axis and the corresponding gains are Routh Horwitz criterion as a theoretical method, and root locus plots as a semi graphical method. A less common one method is that at the intersection points the  $s$  variable has zero real part, and it becomes as  $s=j\omega$ . The new proposed method is to use the derived marginal gain-polynomial of intersection points ready formulas. Then the developed formulas are used to calculate the intersection points and then the corresponding gains.

The comparison is showing the various steps to reach the marginal gain for three examples using the four methods. In the three examples the transfer functions are different in degree and in complexity. The first example is a third order system which has three real poles and no zeros. The second example is a fifth order system which has three real poles, conjugate complex pair poles, and one finite real zero. While the third example is a seventh order system which has three poles at the origin, two conjugate complex pairs of poles, and two conjugate complex pairs of finite zeros.

In the proposed method a polynomial whose roots are the intersection points with the imaginary axis are obtained by using a ready formula. A value substitution of the coefficients of the denominator polynomial, and the coefficients of the numerator's polynomial of the system transfer function into the formula. The polynomial's roots are substituted into the derived ready gain formula which is given in Table 3 to obtain the corresponding gains  $K$ .

The required steps, the amount, and the number of steps of the mathematical operations that are required to reach the final solution for each example are shown in the solution of the three examples. The solution of the three examples using the four methods showed that:

1. For a higher order system, the Routh-Hurwitz criterion is not practical as shown in Example 3. The reason is that the degree of the polynomial of the characteristic equation of a higher order system is high. This makes the calculation of the elements of the Routh array very difficult, and the elements' algebraic expressions are complicated. As a result the Routh-Hurwitz criterion for higher order systems is inapplicable.
2. Using MATLAB to plot the root locus of the control system shows results very close to the actual results, because it is a semi-graphical method, and it increases its inaccuracy when the intersecting points are very close to each other.
3. In the method which is based on the fact that the variable  $s$  on the imaginary axis of the  $s$ -plane has only an imaginary part such as  $s=j\omega$ , There a substitution of  $j\omega$  in the characteristic equation of the control system. The result is an equation

which has a real part and an imaginary part. Those two parts are set to zeros. As a result, we have two equations to be solved to obtain the final polynomial. The roots of the final polynomial are the intersection points. Each value of the intersection points is substituted into one of the equations to be solved to obtain the corresponding gains. The results of this method are the same as the proposed method, but it requires more mathematical steps to reach the results. In addition, in the proposed method the polynomial formula is ready for use while in this method all the solution steps must be done by the user.

4. The proposed method starts from the available transfer function of the control system. Then there is a substitution of the numerical values of the coefficients of the two polynomials, denominator, and the numerator, of the transfer function into the polynomial formula given in Table 4. The substitution gives a polynomial. The roots of this polynomial are substituted in the gain formula which is given in Table 3 to get the marginal gains. This method is accurate, systematic, and it is more efficient than other methods especially for a system of higher order.
5. The result of the proposed method is the same as the results of the other methods. This proves that the proposed method is correct, and accurate. And the solution process is systematic and efficient.
6. The proposed method is applicable for any system order of  $n$ , and it may be programmed and be used to enhance the root locus graph.

It helps in controller tuning if Ziegler and Nichols ultimate cycle criteria is used.

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## APPENDIX

### Mathematical Proof of the Basis of the Proposed Method

In the proposed method a formula is used to construct a polynomial that its roots are the points of the intersection with the imaginary axis where the system has marginal stability. The formula is based on substitution of the coefficients of the two polynomials, denominator, and numerator, of the system transfer function.

From a long division of polynomial, we may write the following equation:

$$\begin{aligned} (\text{Polynomial of Characteristic Equation}, P) &=, \\ &= (\text{Divisor})(\text{Quotient}, Q) + (\text{Remaider 1}, R1) + (\text{Remainder 2}, R2)s. \end{aligned} \quad (\text{A1})$$

### Proof by induction

Starting with  $n=3$  to examine if Equation (A1) is true. To do that the right side of the equation is compared with the left side of the equation. After substitution of the algebraic expressions for each factor of the right side of Equation (A1) that are in Table 1 and Table 2 to give

$$(s^2 + \omega^2)(c_3s + c_2) + (c_0 - c_2\omega^2) + (c_1 - c_3\omega^2)s. \quad (\text{A2})$$

Multiplication and summing of equal terms in Equation (A2) gives

$$c_3s^3 + c_2s^2 + c_3\omega^2s + c_2\omega^2 + c_0 - c_2\omega^2 + c_1s - c_3\omega^2s = c_3s^3 + c_2s^2 + c_1s. \quad (\text{A3})$$

The result as shown in the right side of Equation (A3) is the characteristic polynomial for  $n = 3$ , which means that Equation (A1) is true for this order. Since the remainders are being changed every two successive degrees, as shown in Table 2. It is wise to do the same also for  $n = 4$  to have

$$(s^2 + \omega^2)(c_4s^2 + c_3s + c_2 - c_4\omega^2) + (c_0 - c_2\omega^2 + c_4\omega^4) + (c_1 - c_3\omega^2)s. \quad (\text{A4})$$

Multiplication and summing of equal terms in Equation (A4) give

$$\begin{aligned} c_4s^4 + c_3s^3 + c_2s^2 - c_4\omega^2s^2 + c_4\omega^2s^2 + c_3\omega^2s + c_2\omega^2 - c_4\omega^4 + c_0 - c_2\omega^2 + c_4\omega^4, \\ + c_1s - c_3\omega^2s = c_4s^4 + c_3s^3 + c_2s^2 + c_1s. \end{aligned} \quad (\text{A5})$$

The result is as shown in the right side of Equation (A5) is the characteristic polynomial for  $n = 4$ , which means that Equation (A1) is true for this order.

Assume that Equation (A1) is true for order  $n$  which means that

$$\sum_{i=0,1,2,\dots}^n C_i s^i = (s^2 + \omega^2) \left( \sum_{i=2,3,\dots}^n C_i \sum_{m=2,4,\dots}^i (-1)^{\frac{m}{2}+1} s^{i-m} \omega^{m-2} \right) + \left( \sum_{m=0,2,4,\dots}^n (-1)^{\frac{m}{2}} c_m \omega^m \right) + \left( \sum_{m=1,3,5,\dots}^n (-1)^{\frac{m+3}{2}} c_m \omega^{m-1} \right) s. \quad (A6)$$

Now we must prove that it is also true for  $n + 1$  order which means mathematically that

$$\sum_{i=0,1,2,\dots}^{n+1} C_i s^i = (s^2 + \omega^2) \left( \sum_{i=2,3,\dots}^{n+1} C_i \sum_{m=2,4,\dots}^i (-1)^{\frac{m}{2}+1} s^{i-m} \omega^{m-2} \right) + \left( \sum_{m=0,2,4,\dots}^{n+1} (-1)^{\frac{m}{2}} c_m \omega^m \right) + \left( \sum_{m=1,3,5,\dots}^{n+1} (-1)^{\frac{m+3}{2}} c_m \omega^{m-1} \right) s. \quad (A7)$$

Rewrite Equation (A1) explicitly for degree  $n$  to have

$$\sum_{i=0,1,2,\dots}^n C_i s^i = (s^2 + \omega^2) Q_n + R_{1,n} + R_{2,n} s. \quad (A8)$$

And for degree  $n + 1$  to have

$$\sum_{i=0,1,2,\dots}^{n+1} C_i s^i = (s^2 + \omega^2) Q_{n+1} + R_{1,n+1} + R_{2,n+1} s. \quad (A9)$$

From Tables 1 & 2 we have

$$Q_{n+1} = Q_n + C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \quad (A10)$$

$$R_{1,n+1} = R_{1,n} \quad \text{for even } n,$$

$$R_{1,n+1} = R_{1,n} + (-1)^{\frac{n+1}{2}} c_{n+1} \omega^{n+1} \quad \text{for odd } n. \quad (A11)$$

$$R_{2,n+1} = R_{2,n} + (-1)^{\frac{n+4}{2}} c_{n+1} \omega^n \quad \text{for even } n,$$

$$R_{2,n+1} = R_{2,n} \quad \text{for odd } n. \quad (A12)$$

Substitution of Equations (A11) & (A12) into Equation (A9) for even  $n$  gives

$$\sum_{i=0,1,2,\dots}^{n+1} C_i s^i = (s^2 + \omega^2) \left[ Q_n + C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \right] + R_{1,n} + \left[ R_{2,n} + (-1)^{\frac{n+4}{2}} c_{n+1} \omega^n \right] s. \quad (A13)$$

The last term in the summation in the left side is taken out of the summation and in the right side of the equation the square brackets are opened, and the terms are reordered to have

$$\left( \sum_{i=0,1,2,\dots}^n C_i s^i \right) + C_{n+1} s^{n+1} = (s^2 + \omega^2) Q_n + R_{1,n} + R_{2,n} s + (-1)^{\frac{n+4}{2}} c_{n+1} \omega^n s + (s^2 + \omega^2) \left[ C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \right]. \quad (A14)$$

The right side of Equation (A14) can be expanded as in Equation (A8). So that the first three terms in the right side of Equation (A14) are equal to the first term of the left side of the same equation. So, if those terms are removed from both sides of Equation (A14) we obtain

$$C_{n+1} s^{n+1} = (s^2 + \omega^2) \left[ C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \right] + (-1)^{\frac{n+4}{2}} c_{n+1} \omega^n s. \quad (A15)$$

For even  $n$  the final value of the subscript  $m$  in the summation is equal to  $n$ . In this case the summation term in the right side of Equation (A15) is simplified to be

$$C_{n+1} \sum_{m=2,4,\dots}^n (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} = C_{n+1} (s^{n-1} - s^{n-3} \omega^2 + s^{n-5} \omega^4 - \dots + s^3 \omega^{n-4} - s \omega^{n-2}). \quad (A16)$$

Then the multiplication by the factor  $(s^2 + \omega^2)$  gives

$$C_{n+1} [(s^{n+1} - s^{n-1} \omega^2 + s^{n-3} \omega^4 - \dots + s^5 \omega^{n-4} - s^3 \omega^{n-2}), + (s^{n-1} \omega^2 - s^{n-3} \omega^4 + s^{n-5} \omega^6 - \dots + s^3 \omega^{n-2} - s \omega^n)] = C_{n+1} (s^{n+1} - s \omega^n) \quad (A17)$$

The substitution of the result in Equation (A17) into Equation (A15) gives

$$C_{n+1} s^{n+1} = C_{n+1} (s^{n+1} - s \omega^n) + c_{n+1} \omega^n s \quad (A18)$$

Open parentheses to get

$$C_{n+1}s^{n+1} = C_{n+1}s^{n+1} \tag{A19}$$

So, it is true for even  $n$ .  
For odd  $n$  we have

$$\begin{aligned} \sum_{i=0,1,2,\dots}^{n+1} C_i s^i &= (s^2 + \omega^2) \left[ Q_n + C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \right] +, \\ &+ \left[ R_{1,n} + (-1)^{\frac{n+1}{2}} C_{n+1} \omega^{n+1} \right] + R_{2,n} s. \end{aligned} \tag{A20}$$

Repeating the same steps as for even  $n$  as follows:

Taking out the last term of the summation in the left side of Equation (A20) and open the square brackets in the right side of the equation and reordering the terms to get

$$\begin{aligned} \left( \sum_{i=0,1,2,\dots}^n C_i s^i \right) + C_{n+1} s^{n+1} &= (s^2 + \omega^2) Q_n + R_{1,n} + R_{2,n} s +, \\ &(s^2 + \omega^2) \left[ C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \right] + \left[ (-1)^{\frac{n+1}{2}} C_{n+1} \omega^{n+1} \right]. \end{aligned} \tag{A21}$$

For an odd  $n$  then  $n + 1$  becomes even, and after removing the terms of Equation (A8) from both sides of Equation (A21) we get

$$C_{n+1}s^{n+1} = (s^2 + \omega^2) \left[ C_{n+1} \sum_{m=2,4,\dots}^{n+1} (-1)^{\frac{m}{2}+1} s^{n+1-m} \omega^{m-2} \right] + (-1)^{\frac{n+1}{2}} C_{n+1} \omega^{n+1}. \tag{A22}$$

The summation in Equation (A22) is

$$(s^{n-1} - s^{n-3} \omega^2 + s^{n-5} \omega^4 - \dots, -s^4 \omega^{n-5} + s^2 \omega^{n-3} - \omega^{n-1}) \tag{A23}$$

Then the multiplication by square  $s$  and then by square omega gives

$$\begin{aligned} C_{n+1} [(s^{n+1} - s^{n-1} \omega^2 + s^{n-3} \omega^4 - \dots, -s^6 \omega^{n-5} + s^4 \omega^{n-3} - s^2 \omega^{n-1}), \\ (s^{n-1} \omega^2 - s^{n-3} \omega^4 + s^{n-5} \omega^6 - \dots, -s^4 \omega^{n-3} + s^2 \omega^{n-1} - \omega^{n+1})] &= C_{n+1} (s^{n+1} - \omega^{n+1}). \end{aligned} \tag{A24}$$

Substitute the result of Equation (A24) into Equation (A22) to have

$$C_{n+1}s^{n+1} = C_{n+1}(s^{n+1} - \omega^{n+1}) + C_{n+1}\omega^{n+1} \tag{A25}$$

Or

$$C_{n+1}s^{n+1} = C_{n+1}s^{n+1} \tag{A26}$$

Okay, so it is true for  $n + 1$ .