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Conditions on the Edges and Vertices of Non-commuting Graph

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Graphical abstract

If $\Gamma_G \cong \Gamma_H$ then $|G| = |H|$.

Abstract

Let G be a non-abelian finite group. The non-commuting graph of Γ_G is defined as a graph with a vertex set $G - Z(G)$ in which two vertices x and y are joined if and only if $xy \neq yx$. We define $\Gamma_G = (V(\Gamma_G), E(\Gamma_G))$ such that $V(\Gamma_G)$ is the vertices set and $E(\Gamma_G)$ is the edges set. In this paper, we invest some results on $|E(\Gamma_G)|$, the degree of a vertex of non-commuting graph and the number of conjugacy classes of a finite group. We found that that if $\Gamma_G \cong \Gamma_H$ is a finite group, then $|G| = |H|$.

Keywords: Finite group; non-commuting graph

Abstrak

Katalah G adalah suatu kumpulan terhingga yang bukan abelan. Graf tidak kalis tukar tertib Γ_G ditakrif sebagai graf yang mempunyai set bucu $G - Z(G)$ di mana dua bucu x dan y adalah berkait jika dan hanya jika $xy \neq yx$. Kita takrifkan $\Gamma_G = (V(\Gamma_G), E(\Gamma_G))$ yang mana $V(\Gamma_G)$ adalah set bucu dan $E(\Gamma_G)$ adalah set sisi. Dalam kertas kerja ini, kita hasilkan beberapa keputusan berkaitan $|E(\Gamma_G)|$, iaitu darjah kepada bucu graf tidak kalis tukar tertib dan bilangan kelas konjugat bagi kumpulan terhingga. Kita temui bahawa jika $\Gamma_G \cong \Gamma_H$, dengan H ialah kumpulan terhingga, maka $|G| = |H|$.

Kata kunci: Kumpulan terhingga, graf tidak kalis tukar tertib

1.0 INTRODUCTION

Let G be a non- abelian finite group. Various graphs could be attributed to G, one of which is the non-commuting graph, denoted by Γ_G. The set of vertices and edges of Γ_G are $V(\Gamma_G)$ and $E(\Gamma_G)$ espectively so that $(V(\Gamma_G) = G - Z(G))$ in which $Z(G)$ is the center of G and for every $x, y \in V(\Gamma_G)$ we have:

$$\{x, y\} \in E(\Gamma_G) \Leftrightarrow xy \neq yx.$$

It is apparent that if G is an abelian group, Γ_G would turn to a null graph. For this, G is assumed to be a non-abelian group. The centralizer of x within G, which is denoted by $C_G(x)$, is a subset of G which is defined as $\{g \in G | gx = xg\}$.

Assume that $\Gamma = (V, E)$ is a graph in which V is the set of vertices and E is the set of edges. This graph is assumed to be a finite graph whenever $|V|, |E|$ are finite. The degree of the vertex x which is shown by $deg(x)$ equals to the number of edges through x. According to[4], the non–commuting graph of a finite group G was first introduced by Paul Erdos in connection with the following problem: Let G be a group whose non–commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? By[4] the answer to this question is positive and this was the origin of many similar questions and research. In[1], the relations between some graph properties of Γ_G and the group theory properties of the group G are studied. In particular the following conjecture is raised:

Conjecture 1 Let G be a finite non-abelian group. If there is a group H such that $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

The main purpose of this paper is to put some conditions on $|E(\Gamma_G)|$ of the non–commuting graph so that if $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$. Our notation for graphs is standard and [2] is used as a general reference.

2.0 SOME RESULTS ON CONJUGACY CLASSES

Let G be a finite non-abelian group. The number of conjugacy classes of G is denoted by $k(G)$.

Lemma 2.1[1] Let G be a finite group and $k(G)$ be the number of conjugacy classes of G. Then

$$|E(\Gamma_G)| = 1/2\ |G|\ (|G| - k(G)).$$

Theorem 2.2 Let G and H be finite groups. If $\Gamma_G \cong \Gamma_H$, $(|G|, |H| - k(H)) = 1$ and $(|H|, |G| - k(G)) = 1$, then $|G| = |H|$.

Proof. We have $\Gamma_G \cong \Gamma_H$, thus $|E(\Gamma_G)| = |E(\Gamma_H)|$ and according to assumptions, we can obtain $|G|$ divides $|H|$. Using the same way, $|H|$ divides $|G|$. Therefore, $|G| = |H|$. ■

Theorem 2.3 Let G and H be finite groups. If $\Gamma_G \cong \Gamma_H$ and $k(G) = k(H)$, then $|G| = |H|$.

Proof. We use a contradiction proof. According to the assumptions, it can be written as $|G|^2 - |H|^2 = k(G)(|G| - |H|)$, since $|G| \neq |H|$, thus $k(G) = (|G| + |H|)$. According to the probability of commuting two randomly chosen elements of a finite group G which is equal to $(k(G))/|G|$ .Thus:

$(k(G))/|G| = (|G| + |H|)/|G| = 1 + |H|/|G| > 5/8$.
Based on[3], G is an abelian group and this is a contradiction. Therefore $|G| = |H|$.

3.0 SOME RESULTS ON THE NUMBER OF EDGES

Lemma 3.1 Let G be a finite group. If $|E(\Gamma_G)| = p^n$, where p is a prime number $(p \neq 2)$, then

(i) If n is an even number, then $|G| = 2p^{\frac{n}{2}}$.
(ii) If n is an odd number, then $|G| = p^{\frac{n+1}{2}}$ where $p = 3, 5$.

Proof. Using a contradiction proof, it is shown that $n \neq 1$. There are two cases for $|G|$:

Case 1. If $|G| = 2p$ and $k(G) = 2p - 1$. According to $\frac{k(G)}{|G|} \leq \frac{5}{8}$, the result obtained is $3p \leq 4$ which is a contradiction.
Case 2. If $|G| = p$, then G is abelian and it is a contradiction.
Therefore $n \neq 1$. Now, is is proven that (i) is true, if n is an even number. In this case, there are three forms for $|G|$ which is stated as follows:

Case 1. $|G| = 2p^n$ and $k(G) = 2p^n - 1$. According to [3], $(k(G))/(|G|) \leq 5/8$ and $3p^n \leq 4$. Hence it is impossible for all odd prime number p and all even number n.

Case 2. $|G| = 2p^{n_1}$ and $k(G) = 2p^{n_1} - p^{n_2}$ $(n_1 \geq n_2)$. According to $(k(G))/(|G|) \leq 5/8$, we have $3p^{n_1-n_2} \leq 4$. If $n_1 \neq n_2$, then $3p^{n_1-n_2} > 4$. Thus it is concluded that $n_1 = n_2 . n_1 + n_2 = n$ so $n_2 = n_1 = n/2$ and $|G| = 2p^{\frac{n}{2}}$.

Case 3. $|G| = p^{n_1}$ and $k(G) = p^{n_1} - 2p^{n_2}$, $(n_1 \geq n_2)$. In this case $3p^{n_1-n_2} \leq 16$. If $n_1 = n_2 = n/2$, then $|G| = p^{\frac{n}{2}}$ and $k(G) = -p^{\frac{n}{2}}$ as it is not possible. Using $3p^{n_1-n_2} \leq 16$, we conclude that $n_1 - n_2 = 1$, $p = 3, 5$. Therefore $n_1 = (n+1)/2$ and n_1 cannot be natural number. Hence we have $|G| = p^{\frac{n}{2}}$.
ii) If n is an odd number, then there exist three cases for $|G|$:

Case 1. $|G| = 2p^n$ and $k(G) = 2p^n - 1$. It is not possible for all odd prime numbers p and all odd numbers n.

Case 2. $|G| = 2p^{n_1}$ and $k(G) = 2p^{n_1} - p^{n_2}$, $(n_1 \geq n_2)$. We have $(k(G))/(|G|) \leq 5/8$, therefore $3p^{n_1-n_2} \leq 4$. If $n_1 \neq n_2$, then $3p^{n_1-n_2} > 4$. It follows that$n_1 = n_2$. Hence $n_1 = n_2 = n/2$. Since n is an odd number, n_1 can not be natural number. Therefore, this case is impossible.

Case 3. $|G| = p^{n_1}$ and $k(G) = p^{n_1} - 2p^{n_2}$, $(n_1 \geq n_2)$. We will gain $n_1 - n_2 = 1$, $p = 3{,}5$. In this case, $n_1 = (n+1)/2$, $n_2 = (n-1)/2$ and $|G| = 3^{\frac{n+1}{2}}$ or $5^{\frac{n+1}{2}}$. ■

Theorem 3.2 Let G and H be finite non–abelian groups. If $\Gamma_G \cong \Gamma_H$ and $|E(\Gamma_G)| = p^n$ (p is an odd prime number) then $|G| = |H|$.
Proof. This result can be proven easily by Lemma 3.1.

Lemma 3.3 Let G be a finite non-abelian group. If $|E(\Gamma_G)| = 2^n$ and n is an even number, then $|G| = 2^{\frac{n}{2}+1}$.

Proof. We have $|E(\Gamma_G)| = 2^n$ then $|G| = 2^{n_1}$ and $k(G) = 2^{n_1} - 2^{n_2}$ as $n_1 + n_2 = n + 1$ and $n_1 \geq n_2$. Using [3] we will have $3.2^{n_1} \leq 2^{n_2+3}$. Therefore, $n_1 = n_2 + 1$ *or* $n_1 = n_2 + 2$. If $n_1 = n_2 + 2$, then $3.2^{n_2+2} \leq 2^{n_2+3}$. Therefore $3 \leq 2$ and it is a contradiction. Thus $n_1 = n_2 + 1$ and on the other hand $n_1 + n_2 = n + 1$ and it is concluded that $n_2 = \frac{n}{2}, n_1 = \frac{n}{2} + 1$. As a result $|G| = 2^{\frac{n}{2}+1}$. ■

Theorem 3.4 Let G be a finite group. If H is a group, $\Gamma_G \cong \Gamma_H$ and $|E(\Gamma_G)| = 2^n$ (n is an even number), then $|G| = |H|$.

Proof. It follows from Lemma 3.3.

Lemma 3.5 Let G be a finite group. If $|E(\Gamma_G)| = p^2q$ (p, q are prime numbers and $p > q$), then $|G| = 3p$ or $5p$.

Proof. $2p^2q = |G|(|G| - k(G))$ is resulted by $|E(\Gamma_G)| = \frac{1}{2}|G|(|G| - k(G))$ and $|G| = 2p^2, 2q, p^2q, 2pq, pq$ or $2p$. Now we investigate all cases:
Case 1. If $|G| = 2p^2$, then $k(G) = 2p^2 - q$. According to $\frac{k(G)}{|G|} \leq \frac{5}{8}$, we have $3p^2 \leq 4q$ hence $|G| \neq 2p^2$.
Case 2. If $|G| = 2q$, then $k(G) = 2q - p^2 < 0$. Hence $|G| \neq 2q$.

Case 3. If $|G| = p^2q$, then $k(G) = p^2q - 2$. This resulted as $3p^2q \leq 16$. There are not any two prime numbers that satisfy this inequality, thus $|G| \neq p^2q$.

Case 4. If $|G| = 2pq$ then $k(G) = 2pq - p$. We obtain $3q \leq 4$, and this is impossible.

Case 5. If $|G| = pq$, then $k(G) = pq - 2p$ and $3q \leq 16$. q can be 2, 3 or 5. If $q = 2$ then $k(G) = 0$ so $|G| = 3p$ or $5p$.

Case 6. If $|G| = 2p$ then $k(G) = 2p - pq \leq 0$. That is not possible. So $|G| \neq 2p$.

Using results in Lemma 3.5, we provide the following theorem:

Theorem 3.6 Let G and H be finite groups. If $\Gamma_G \cong \Gamma_H$ and $|E(\Gamma_G)| = p^2q$ (where p and q are prime numbers, $p > q$) then $|G| = |H|$.

Proof. Using recent lemma, we have $|G| = 3p$ or $5p$. Without loss of generality, suppose that $|G| = 3p$, so prove that $|H| = 3p$.
Suppose that $|H| = 5p$. We know that $\Gamma_G \cong \Gamma_H$ then $|V(\Gamma_G)| = |V(\Gamma_H)|$. It means $|G| - |Z(G)| = |H| - |Z(H)|$, there are three cases for $|Z(G)|$:

Case 1. If $|Z(G)| = 1$, then $|Z(H)| = 2p - 1$ and $2p - 1 \mid |Z(H)| = 5p$, this occurs when $p = 3$. Therefore, G is an abelian group and $G = Z(G)$. That is impossible. In this case $|G| = |H| = 3p$.

Case 2. If $|Z(G)| = 3$, then $|Z(H)| = 2p - 3$. It occurs when $p = 3$. Thus $|G| = |H| = 3p$.

Case 3. If $|Z(G)| = p$, then $|Z(H)| = 3p$ and $|Z(H)| \nmid 5p$, hence $|G| = |H| = 3p$.

Respectively, we can show that if $|G| = 5p$, then $|H| = 5p$. ■

Theorem 3.7 There is no finite group that the number of edges of its non–commuting graph be $2p$, where p is an odd prime.

Proof. Suppose that G is a finite group and $|E(\Gamma_G)| = 2p$. We have $4p = |G|(|G| - k(G))$, then $|G| = 4p$ or $|G| = 2p$.

If $|G| = 4p$, then $k(G) = 4p - 1$. Using $\frac{k(G)}{|G|} \leq \frac{5}{8}$, it is obtained that $3p \leq 2$. This not true for all odd prime numbers. Now, if $|G| = 2p$ then $k(G) = 2p - 2$. Using $\frac{k(G)}{|G|} \leq \frac{5}{8}$, we will have $3p \leq 8$. Again, this not true for all odd prime numbers. We conclude that, there is no such group.

■4.0 DEGREE OF A VERTEX OF NON–COMMUTING GRAPH

Lemma 4.11 Let G be a finite group. If x is one of the vertices of Γ_G, then

$$deg(x) = |G| - |C_G(x)|.$$

Theorem 4.2 Let G be a finite group such that there is an element $g \in G - Z(G)$ with $deg(g) = p^2q$, where p and q are prime numbers. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

Proof. From $|C_G(g)|\left(\frac{|G|}{|C_G(g)|} - 1\right) = p^2q$ we deduced that $|C_G(g)| = p, p^2, q, pq$ and p^2q , hence
$|G| = p(pq + 1), p^2(q + 1), q(p^2 + 1), pq(p + 1)$ and $2p^2q$
Since the corresponding element $g' \in H - Z(H)$ has also degree p^2q we will obtain
$|H| = p(pq + 1), p^2(q + 1), q(p^2 + 1), pq(p + 1)$ and $2p^2q$. We use contradiction to show $|G| = |H|$.
Since $|G| = p(pq + 1)$ and $|G| \neq |H|$, then there exists four forms for $|H|$:

1. From $|G| = p(pq + 1)$ we obtain $|C_G(g)| = p$, hence $|Z(G)| = 1$. If $|H| = p^2(q + 1)$ and since $\Gamma_G \cong \Gamma_H$, we have $|Z(H)| = p^2 - p + 1$. Therefore $|C_G(g')| = p^2$ and $|Z(H)| \nmid |C_G(g')|$. This case is impossible.

2. If $|H| = q(p^2 + 1)$, where $|C_G(g')| = q$. Using this equality $|G| - |Z(G)| = |H| - |Z(H)|$ thus, $|Z(H)| = q - p + 1$. The order of $Z(H)$ must divide $|C_G(g')|$. It means $(q - p + 1) \mid q$. This is impossible.

3. If $|H| = pq(p + 1)$, we must have $|C_G(g')| = pq$. In this case $|Z(H)| = pq - p + 1$ and since the $|Z(H)| \mid |C_G(g')|$, there is three cases for $|Z(H)|$:

Case 1. If $|Z(H)| = p = pq - p + 1$, then $p(q - 1) = p - 1$. It is not possible.

Case 2. If $|Z(H)| = q = pq - p + 1$, then $p = 1$. It is contradiction.
Case 3. $|Z(H)| \neq 1$. It is clear.
If $|G| = p(pq + 1)$, then $|H| = p(pq + 1)$. It means $|G| = |H|$.

Simply, we can consider different scenarios to reach the desired result. ■

Theorem 4.3 Let G be a finite group such that there is an element $g \in G - Z(G)$ with $deg(g) = p^2q^2$, where p and q are prime numbers. If H is a group and $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

Proof. From $|C_G(g)|\left(\frac{|G|}{|C_G(g)|} - 1\right) = p^2q^2$ we have
$|C_G(g)| = p, p^2, q, q^2, pq, p^2q, pq^2$ and p^2q^2. Respectively
$|G| = p(pq^2 + 1), p^2(q^2 + 1), q(p^2q + 1), q^2(p^2 + 1), pq(pq + 1), p^2q(q + 1), pq^2(p + 1)$ and $2p^2q^2$.
Since the corresponding element $g' \in H - Z(H)$ has also degree p^2q^2, we will obtain

$|H| = p(pq^2 + 1), p^2(q^2 + 1), q(p^2q + 1), q^2(p^2 + 1), pq(pq + 1), p^2q(q + 1), pq^2(p + 1)$ and $2p^2q^2$. Without loss of generality, assume that $|G| = 2p^2q^2$, from $|G|$ we obtain $|C_G(g)| = p^2q^2$ and since $|G| \neq |H|$, there exists seven cases for $|H|$ stated as follows:

1. If $|H| = p(pq^2 + 1)$, we gain $|Z(H)| = 1$. Using of this equality $|G| - |Z(G)| = |H| - |Z(H)|$, thus $|Z(G)| = p^2q^2 - p + 1$. It is impossible, since $(p^2q^2 - p + 1) \nmid p^2q^2$.

2. If $|H| = p^2(q^2 + 1)$, then $|Z(H)| = 1$ or p. If $|Z(H)| = 1$, then $|Z(G)| = p^2q^2 - p^2 + 1$. This is not true since $(p^2q^2 - p^2 + 1) \nmid p^2q^2$. If $|Z(H)| = p$, then $|Z(G)| =$

$p^2q^2 - p^2 + p$, but we have $(p^2q^2 - p^2 + p) \nmid p^2q^2$. Therefore $|H| \neq p^2(q^2+1)$.

3. If $|H| = q(p^2q+1)$, we have $|Z(H)| = 1$. Using the equality $|G| - |Z(G)| = |H| - |Z(H)|$, $|Z(G)| = (p^2q^2 - q + 1)$. It is impossible, because $(p^2q^2 - q + 1) \nmid p^2q^2$.

4. If $|H| = q^2(p^2+1)$, then $|Z(H)| = 1, q$. If $|Z(H)| = 1$, we have $|Z(G)| = (p^2q^2 - q^2 + 1)$ and p^2q^2 is not divisible by $(p^2q^2 - q^2 + 1)$. Now, assume that $|Z(H)| = q$, in this case $|Z(G)| = (p^2q^2 - q^2 + q)$. Again it is not true.

5. If $|H| = pq(pq+1)$, then $|Z(H)| = 1, q$ or p. Clearly, this is not true.

6. If $|H| = p^2q\,(q+1)$, then $|Z(H)| = 1, p, p^2, q$ or pq.
   If $|Z(H)| = 1$, then $|Z(G)| = (p^2q^2 - p^2q + 1)$.
   If $|Z(H)| = p$, then $|Z(G)| = (p^2q^2 - p^2q + p)$.
   If $|Z(H)| = q$, then $|Z(G)| = (p^2q^2 - p^2q + q)$.
   If $|Z(H)| = p^2$, then $|Z(G)| = (p^2q^2 - p^2q + p^2)$.
   If $|Z(H)| = pq$, then $|Z(G)| = (p^2q^2 - p^2q + pq)$.

All of the above are impossible, because $|Z(G)| \nmid p^2q^2$ for all mentioned cases.

7. If $|H| = pq^2(p+1)$, then $|Z(H)| = 1, p, q^2, q$ or pq. As in 6 it is not true.

Therefore, $|G| = |H|$.

5.0 CONCLUSION

One of the important graphs that could be attributed to G is non-commuting graph. It defines as a graph with a vertex set $G - Z(G)$ in which two vertices x and y are joined if and only if $xy \neq yx$. In introduction, we mentioned two conjectures. In this research, we put some conditions on the number of edges set and degree vertices so that the conjectures become true.

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