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Numerical Simulation of a Dynamic Contact Issue for Piezoelectric Materials

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Abstract: This paper deals with the numerical analysis of a dynamic contact problem with friction between a piezoelectric body and an electrically conductive foundation. The material's behaviour is described by means of an electro-viscoelastic constitutive law. The contact is modelled using the classical normal damped response condition and a friction law. The mechanical model is described as a coupled system of a nonlinear variational equation for the velocity field and a linear variational equation for the electric potential field. The discrete scheme of the coupled system is introduced based on a finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivate. The frictional contact is treated by using a penalized approach and a version of Newton's method. A solution algorithm is discussed and implemented. Finally, numerical simulation results are reported on a two-dimensional test problem. These simulations show the performance of the algorithm and illustrate the effects of the conductivity of the foundation, as well.

Keywords: Dynamic process; Finite element; Normal damped response; Numerical simulations; Piezoelectric material.

1. INTRODUCTION

Piezoelectrics are materials that can create electricity when subjected to a mechanical stress. They will also work in reverse, generating a strain by the application of an electric field. Piezoelectric materials present a great importance in the development hight technological applications such as actuators, sensors, engineering control equipment's or smart materials and structures, because of the coupling effects between mechanical and electric fields. During the last ten years, numerous contact problems involving this piezoelectric effect have been studied from the variational and numerical points of view (see for example [1-5] and more recently [6-10]).

Dynamic frictional contact problems for electro-viscoelastic materials were studied in [11, 12], where the contact is modelled with a general normal damped response condition and a friction law, which are nonmonotone, possibly multivalued and have the subdifferential from. In [11] the foundation was assumed insulated and in [12] it was assumed to be electrically conductive and the electrical condition on the contact surface is modelled with subdifferential boundary condition. A similar model was studied recently in [13] in the case of a non-clamped body and the Dirichlet boundary condition is not assumed at the part of body's surface. The results in [13] concern mainly the numerical analysis of the problem while the results in [11, 12] concern the existence of solutions for problems and are obtained by using arguments of hemivariational inequalities, however, no numerical analysis was studied.

The current work represents a continuation of [11, 12] and it deals with a numerical modelling and the numerical simulation of contact between a body and a conductive foundation. The body is assumed to be viscoelastic and piezoelectric and the contact is dynamic. Unlike [11, 12], the contact is modelled by a classical normal damped response condition, in which the response of foundation depends on the speed (see [14]). The associated frictional law is also included; as a consequence, the resulting variational formulation of the problem is different from that in [11 , 12] and represents a new mathematical model, which is in a form of a system coupling a nonlinear variational equation for the velocity field with a time-dependent linear equation for the electric potential field. The other trait of novelty of the present paper consists in the fact that here we deal with the numerical approach of the problem and provide numerical simulations. The corresponding numerical scheme is based on the spatial and temporal discretization. Furthermore, the spatial discretization is based on the finite element method, while the temporal discretization is based on the Euler scheme. Then, the scheme was utilized as a basis of a numerical code for the problem. By using the code, simulation results on numerical example are presented.

Following this introduction, the rest of the paper is structured as follows. In Section 2 the mechanical model is presented together with its variational formulation. A fully discrete scheme is presented in Section 3, and the numerical treatment of the frictional contact conditions is realized using a specific penalized approach and a version of Newton's method. In Section 4, some numerical simulations are presented to highlight the performance of the method and the effects of the conductivity of the foundation, as well. Conclusions are finally drawn in section 5.

2. PROBLEM STATEMENT

Consider a body made of a piezoelectric material occupies the domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ with a smooth boundary $\partial\Omega = \Gamma$. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we consider a partition of Γ into three open disjoint parts Γ_D, Γ_N, Γ_C, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\Gamma_D > 0$ and $meas\Gamma_a > 0$. The body is clamped on Γ_D and therefore the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_N$ act on Γ_N. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_b is prescribed on Γ_b . In the reference configuration, the body is in contact over Γ_C with a conductive obstacle, the so-called foundation. We assume that the foundation is electrically conductive and its potential is maintained at φ_f . The contact is frictional and there may be electrical charges on the contact surface.

We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Also, below $\boldsymbol{\nu} = (\nu_i)$ represents the unit outward normal on Γ while " **.** " and ‖ **.** ‖ denote the inner product and the Euclidean norm on $\mathbb{R}^d$ and S^d, respectively, that is $\boldsymbol{u}.\boldsymbol{v} = u_i v_i$, $||\boldsymbol{v}|| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$ for $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^d$ and $\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $||\boldsymbol{\tau}|| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S^d$. Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, i.e. $f_{,i} = \frac{\partial f}{\partial x_i}$.

The mechanical problem of dynamic frictional contact of the electro-viscoelastic body with damped responses may be formulated as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \longrightarrow \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \longrightarrow S^d$, *an electric potential* $\varphi : \Omega \times [0, T] \longrightarrow \mathbb{R}$ *and an electric displacement field* $\boldsymbol{D} : \Omega \times [0, T] \longrightarrow \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{B}\varepsilon(\boldsymbol{u}) - \mathcal{E}^T\boldsymbol{E}(\varphi) \quad in\ \Omega \times (0, T), \tag{1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \beta\boldsymbol{E}(\varphi) \quad in\ \Omega \times (0, T), \tag{2}$$

$$Div\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \rho\, \ddot{\boldsymbol{u}} \quad in\ \Omega \times (0, T), \tag{3}$$

$$div\, \boldsymbol{D} = q_0 \quad in\ \Omega \times (0, T), \tag{4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad on\ \Gamma_D \times (0, T), \tag{5}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad on\ \Gamma_N \times (0, T), \tag{6}$$

$$\varphi = 0 \quad on\ \Gamma_a \times (0, T), \tag{7}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = q_b \quad on\ \Gamma_b \times (0, T), \tag{8}$$

$$-\sigma_\nu = p_\nu(\dot{u}_\nu), \qquad -\boldsymbol{\sigma}_\tau = p_\tau(\dot{\boldsymbol{u}}_\tau) \quad on\ \Gamma_C \times (0, T), \tag{9}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = k_e(\varphi - \varphi_f) \quad on\ \Gamma_C \times (0, T), \tag{10}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad in\ \Omega. \tag{11}$$

In Equations (1) - (11), in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$ and the time variable $t \in [0, T]$, where $T > 0$.

Equations (1) and (2) represent the electro-viscoelastic constitutive law of the material in which denotes $\boldsymbol{\sigma} = (\sigma_{ij})$ the stress tensor, $\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi)$ is the electric field. $\mathcal{A} = (a_{ijkl})$, $\mathcal{B} = (b_{ijkl})$, $\mathcal{E} = (e_{ijk})$ and $\beta = (\beta_{ij})$ are respectively, the viscosity, elasticity, piezoelectric and permittivity tensors. $\mathcal{E}^T$ is the transpose of $\mathcal{E}$. We recall that $\varepsilon_{ij}(\boldsymbol{u})) = (u_{i,j} + u_{j,i})/2$ and $\boldsymbol{E}(\varphi) = -\nabla\varphi = -(\varphi_{,i})$. Also the tensors $\mathcal{E}$ and $\mathcal{E}^T$ satisfy the equality $\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^T\boldsymbol{v} \ \ \forall\, \boldsymbol{\sigma} \in S^d$, $\boldsymbol{v} \in \mathbb{R}^d$, and the components of the tensor $\mathcal{E}^T$ are given by $e^T_{ijk} = e_{kij}$.

Equation (3) is the equation of motion in which ρ denotes the constant mass density and Equation (4) represents the balance equation for the electric displacement field, in which " Div " and " div " denote the divergence operators for tensor and vector valued functions, i.e. $Div\, \boldsymbol{\sigma} = (\sigma_{ij,j})$, $div\, \boldsymbol{D} = (D_{i,i})$. We use these equations since the process is assumed to be mechanically dynamic and electrically static.

Equations (5) and (6) are the displacement and traction boundary conditions, whereas Equations (7) and (8) represent the electric boundary conditions; these conditions model the fact that the displacement field and the electrical potential vanish on Γ_D and Γ_a, respectively, while the forces and the electric charges are prescribed on Γ_N and Γ_b respectively.

We turn to describe the contact. Since we assume the foundation to be deformable and a normal damped response contact condition is used (see [14, 15]), which means that the foundation is reactive. Thus, the normal stress $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$ on the contact surface is obtained through the following semilinear relation $-\sigma_\nu = p_\nu(\dot{u}_\nu)$, where $\dot{u}_\nu = \dot{\boldsymbol{u}}\cdot\boldsymbol{\nu}$ denotes the normal velocity. The normal damped response function p_ν is prescribed and satisfies $p_\nu(\dot{u}_\nu)=0$ for $\dot{u}_\nu \le 0$, since then there is no contact. As an example, we may consider (see [15]),

$$p_\nu(r) = c_\nu r_+ \,, \tag{12}$$

where $c_\nu > 0$ is a positive constant which represents a deformability coefficient and $r_+ = \max\{r, 0\}$. Moreover, we consider the following associated friction law $-\boldsymbol{\sigma}_\tau = p_\tau(\dot{\boldsymbol{u}}_\tau)$. Here, $\dot{\boldsymbol{u}}_\tau = \dot{\boldsymbol{u}} - \dot{u}_\nu\boldsymbol{\nu}$ is the tangential velocity, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ denotes the shear stresses and p_τ is a constitutive function whose properties will be described below. In the numerical simulations presented in Section 4, the following form of p_τ was considered

$$p_\tau(\boldsymbol{r}) = c_\tau \boldsymbol{r} \quad \forall\, \boldsymbol{r} \in \mathbb{R}^d. \tag{13}$$

In this case, $c_\tau > 0$ is a frictional coefficient and the tangential shear is proportional to the tangential velocity. Equation (10) is the associated electric boundary condition on Γ_C, where k_e is a positive constant, the electrical conductivity coefficient. Finally, Equation (11) represent the initial conditions where $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ denote the initial displacement and the initial velocity, respectively.

To present the variational formulation of Problem P we need some additional notation and preliminaries. We start by introducing the spaces $H = L^2(\Omega, \mathbb{R}^d)$, $\mathcal{K} = L^2(\Omega, S^d)$. The spaces H and $\mathcal{K}$ are Hilbert spaces equipped with the inner products $(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx$ and $(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{K}} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\;dx$, respectively. The associated norms in H and $\mathcal{K}$ are denoted by $||\cdot||_H$ and $||\cdot||_{\mathcal{K}}$, respectively.

For the displacement and the electric potential fields, we introduce the spaces $V = \{\boldsymbol{v} \in H^1(\Omega, \mathbb{R}^d);\ \boldsymbol{v} = \boldsymbol{0}\ \ on\ \ \Gamma_D\}$ and $W = \{\psi \in H^1(\Omega);\ \psi = 0\ \ on\ \ \Gamma_a\}$. On V and W we consider the inner products and the corresponding norms given by

$$(\boldsymbol{u},\boldsymbol{v})_V = \big(\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v})\big)_{\mathcal{K}}\,,\quad ||\boldsymbol{v}||_V = ||\varepsilon(\boldsymbol{v})||_{\mathcal{K}} \ \text{ for all } \boldsymbol{u},\boldsymbol{v}\in V, \tag{14}$$

$$(\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_H\,,\quad ||\psi||_W = ||\nabla\psi||_H \ \text{ for all } \varphi,\psi \in W. \tag{15}$$

Since $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$ are positive, it follows from the Korn and the Friedrichs-Poincaré inequalities, respectively, that $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ are Hilbert spaces.

We consider the four mapping $j : V \times V \longrightarrow \mathbb{R}$, $J : W\times W \longrightarrow \mathbb{R}$, $\boldsymbol{f} : [0,T] \longrightarrow V$ *and* $q : [0,T] \longrightarrow W$, defined by

$$j(\boldsymbol{v},\boldsymbol{w}) = \int_{\Gamma_C} p_\nu(v_\nu) w_\nu\, da + \int_{\Gamma_C} p_\tau(\boldsymbol{v}_\tau)\cdot\boldsymbol{w}_\tau\, da, \tag{16}$$

$$J(\varphi,\psi) = \int_{\Gamma_C} k_e\big(\varphi - \varphi_f\big)\psi\, da\,, \tag{17}$$

$$(\boldsymbol{f}(t),\boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{w}\,dx + \int_{\Gamma_N} \boldsymbol{f}_N(t)\cdot\boldsymbol{w}\,da, \tag{18}$$

$$(q(t),\psi)_W = \int_\Omega q_0(t)\psi\,dx + \int_{\Gamma_b} q_b(t)\psi\,da, \tag{19}$$

for all $\boldsymbol{w} \in V$ and $\psi \in W$.

Then, performing integration by parts, we obtain the following variational formulation of Problem P in terms of the velocity field $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$ and the electric potential fields.

Problem P_V. *Find a velocity field* $\boldsymbol{v} : [0,T] \longrightarrow V$ *and an electric potential* $\varphi : [0,T] \longrightarrow W$ *such that* $\boldsymbol{v}(0) = \boldsymbol{v}_0$ *and for a.e.* $t \in (0,T)$

$$\begin{aligned}(\rho\dot{\boldsymbol{v}}(t),\boldsymbol{w})_{\boldsymbol{H}} + \Big(\mathcal{A}\varepsilon\big(\boldsymbol{v}(t)\big),\varepsilon(\boldsymbol{w})\Big)_{\mathcal{K}} + \Big(\mathcal{B}\varepsilon\big(\boldsymbol{u}(t)\big),\varepsilon(\boldsymbol{w})\Big)_{\mathcal{K}} + \big(\mathcal{E}^T\nabla\varphi(t),\varepsilon(\boldsymbol{w})\big)_{\mathcal{K}}\\ +\ j(\boldsymbol{v}(t),\boldsymbol{w}) = (\boldsymbol{f}(t),\boldsymbol{w})_V \qquad \forall\,\boldsymbol{w}\in V,\end{aligned} \tag{20}$$

$$(\beta\nabla\varphi(t),\nabla\psi)_H - \big(\mathcal{E}\varepsilon\big(\boldsymbol{u}(t)\big),\nabla\psi\big)_H + J(\varphi(t),\psi) = (q(t),\psi)_W \qquad \forall\psi\in W, \tag{21}$$

where the displacement field $u(t)$ *is then defined as* $\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,ds + \boldsymbol{u}_0$.

3. NUMERICAL APPROACH

The discretization of P_V will be done in two steps. First, we consider two finite dimensional spaces $V^h \subset V$ and $W^h \subset W$ approximating the spaces V and W, respectively. $h > 0$ denotes the spatial discretization parameter. Secondly, the time derivatives are discretized by using a uniform partition of $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N \le T$. Let k be the time step size, $k = T/N$, and for a continuous function $f(t)$ let $f_n = f(t_n)$. Finally, for a sequence $\{w_n\}_{n=0}^N$ we denote by $\delta w_n = (w_n - w_{n-1})/k$ the divided differences.

The fully discrete approximation of Problem P_V, based on the forward Euler scheme, is the following.

Problem P_V^{hk}. *Find a discrete velocity field* $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete electric potential* $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h$

such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$ *and for all* $n = 1 \dots N$,

$$(\rho\delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \left(\mathcal{A}\varepsilon(\boldsymbol{v}_n^{hk}), \varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}} + \left(\mathcal{B}\varepsilon(\boldsymbol{u}_n^{hk}), \varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}} + \left(\mathcal{E}^T\nabla\varphi_n^{hk}, \varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}} + j(\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h) = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h, \tag{22}$$

$$(\beta\nabla\varphi_n^{hk}, \nabla\psi^h)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_n^{hk}), \nabla\psi^h)_H + J(\varphi_n^{hk}, \psi^h) = (q_n, \psi^h)_W \quad \forall \psi^h \in W^h, \tag{23}$$

where the displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *is given by* $\boldsymbol{u}_n^{hk} = k\sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h$.

Here $\boldsymbol{u}_0^h$ and $\boldsymbol{v}_0^h$ are appropriate approximation of the initial condition $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$, respectively, and φ_0^{hk} is the unique solution of the second equation in Problem P_V^{hk} for $n = 0$.

We notice that the fully discrete problem P_V^{hk} can be seen as a coupled system of variational equations. Using classical results of nonlinear variational equations (see [16]), we obtain that Problem P_V^{hk} admits a unique solution $\boldsymbol{v}^{hk} \subset V^h$ and $\varphi^{hk} \subset W^h$.

The algorithm, used in solving the fully discrete frictional contact problem P_V^{hk}, is based on a backward Euler difference for the time derivatives and on a penalty approach to simulate the frictional contact conditions. For more considerations about Computational Contact Mechanics, see the monographs [17, 18, 19].

Let N_{tot} be the total number of nodes and denote by α^i, γ^i the basis functions of the space V^h and W^h, respectively, for $i = 1, \dots, N_{tot}$. Then the expressions of the functions $\boldsymbol{w}^h$ and ψ^h is given by $\boldsymbol{w}^h = \sum_{i=1}^{N_{tot}} \boldsymbol{w}^i\alpha^i$, $\psi^h = \sum_{i=1}^{N_{tot}} \psi^i\gamma^i$ where $\boldsymbol{w}^i$ and ψ^i represent the values of the corresponding functions $\boldsymbol{w}$ and ψ at the i^{th} node of the uniform triangulation of Ω, denoted by T^h.

The penalized approach we use shows that the Problem P_V^{hk} can be governed by the following system of nonlinear equations

$$R(\delta\boldsymbol{v}_n, \boldsymbol{v}_n, \boldsymbol{u}_n, \varphi_n) = \widetilde{M}(\delta\boldsymbol{v}_n) + \tilde{A}(\boldsymbol{v}_n) + \mathrm{G}(\boldsymbol{u}_n, \varphi_n) + F(\boldsymbol{v}_n, \varphi_n) = \boldsymbol{0}, \tag{24}$$

that we describe below. First, the vectors $\delta\boldsymbol{v}_n$, $\boldsymbol{v}_n$, $\boldsymbol{u}_n$ and φ_n represent the acceleration, velocity, displacement and electric potential generalized vectors, respectively, defined by $\delta\boldsymbol{v}_n = \{\delta\boldsymbol{v}_n^i\}_{i=1}^{N_{tot}}$, $\boldsymbol{v}_n = \{\boldsymbol{v}_n^i\}_{i=1}^{N_{tot}}$, $\boldsymbol{u}_n = \{\boldsymbol{u}_n^i\}_{i=1}^{N_{tot}}$ and $\varphi_n = \{\varphi_n^i\}_{i=1}^{N_{tot}}$, where $\delta\boldsymbol{v}_n^i := \frac{\boldsymbol{v}_n^i - \boldsymbol{v}_{n-1}^i}{k}$, $\boldsymbol{v}_n^i := \frac{\boldsymbol{u}_n^i - \boldsymbol{u}_{n-1}^i}{k}$, $\boldsymbol{u}_n^i$ and φ_n^i represent the values of the corresponding functions $\delta\boldsymbol{v}_n^{hk}$, $\boldsymbol{v}_n^{hk}$, $\boldsymbol{u}_n^{hk}$ and φ_n^{hk} at the i^{th} node of T^h.

The specific penalized contact operator $F(\boldsymbol{v}_n, \varphi_n)$, which permits to take into account the conductivity of the foundation, is given by

$$\left(F(\boldsymbol{v}_n, \varphi_n), (\boldsymbol{w}, \psi)\right)_{\mathbb{R}^{d\times N_{tot}}\times\mathbb{R}^{N_{tot}}} = j(\boldsymbol{v}_n, \boldsymbol{w}) + J(\varphi_n, \psi) \quad \forall\, \boldsymbol{w} \in \mathbb{R}^{d\times N_{tot}}, \forall\, \psi \in \mathbb{R}^{N_{tot}}. \tag{25}$$

In addition, the generalized acceleration term $\widetilde{M}(\delta\boldsymbol{v}_n) \in \mathbb{R}^{d\times N_{tot}} \times \mathbb{R}^{N_{tot}}$ and the generalized viscous term $\tilde{A}(\boldsymbol{v}_n) \in \mathbb{R}^{d\times N_{tot}} \times \mathbb{R}^{N_{tot}}$ are defined by $\widetilde{M}(\delta\boldsymbol{v}_n) = \left(M(\delta\boldsymbol{v}_n), \boldsymbol{0}_{N_{tot}}\right)$ and $\tilde{A}(\boldsymbol{v}_n) = (A(\boldsymbol{v}_n), \boldsymbol{0}_{N_{tot}})$. Here, $\boldsymbol{0}_{N_{tot}}$ is the zero element of $\mathbb{R}^{N_{tot}}$ and $M(\delta\boldsymbol{v}_n) \in \mathbb{R}^{d\times N_{tot}}$, $A(\boldsymbol{v}_n) \in \mathbb{R}^{d\times N_{tot}}$, $G(\boldsymbol{u}_n, \varphi_n) \in \mathbb{R}^{d\times N_{tot}} \times \mathbb{R}^{N_{tot}}$ denote the acceleration term, the viscous term and the elastic-piezoelectric term, respectively, given by

$$(M(\delta\boldsymbol{v}_n)\cdot\boldsymbol{w})_{\mathbb{R}^{d\times N_{tot}}} = (\rho\delta\boldsymbol{v}_n, \boldsymbol{w}^h)_H \quad \forall\, \boldsymbol{w} \in \mathbb{R}^{d\times N_{tot}},\ \forall\, \boldsymbol{w}^h \in V^h, \tag{26}$$

$$(A(\boldsymbol{v}_n)\cdot\boldsymbol{w})_{\mathbb{R}^{d\times N_{tot}}} = \left(\mathcal{A}\varepsilon(\boldsymbol{v}_n), \varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}} \quad \forall\, \boldsymbol{w} \in \mathbb{R}^{d\times N_{tot}},\ \forall\, \boldsymbol{w}^h \in V^h, \tag{27}$$

$$\begin{aligned}\left(G(\boldsymbol{u}_n, \varphi_n)\cdot(\boldsymbol{w}, \psi)\right)_{\mathbb{R}^{d\times N_{tot}}\times\mathbb{R}^{N_{tot}}} &= \left(\mathcal{B}\varepsilon(\boldsymbol{u}_n^{hk}), \varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}} + \left(\mathcal{E}^T\nabla\varphi_n^{hk}, \varepsilon(\boldsymbol{w}^h)\right)_{\mathcal{K}} - (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \\ &+ (\beta\nabla\varphi_n^{hk}, \nabla\psi^h)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_n^{hk}), \nabla\psi^h)_H - (q_n, \psi^h)_W \\ &\forall\, \boldsymbol{w} \in \mathbb{R}^{d\times N_{tot}}, \quad \forall\, \psi \in \mathbb{R}^{N_{tot}}, \quad \forall\, \boldsymbol{w}^h \in V^h, \quad \forall \psi^h \in W^h.\end{aligned} \tag{28}$$

Above, $\boldsymbol{w}$ and ψ represent the generalized vector of components $\boldsymbol{w}^i$ and ψ^i for $i = 1, \dots, N_{tot}$, respectively, and note that the volume and surface efforts are contained in the term $\mathrm{G}(\boldsymbol{u}_n, \varphi_n)$.

The solution algorithm consists in a combination between the finite differences (backward Euler difference) and the linear iterations methods (Newton method). The finite difference scheme we use is characterized by a first order time integration scheme, both for the acceleration $\delta\boldsymbol{v}_n$ and the velocity $\boldsymbol{v}_n = \delta\boldsymbol{u}_n$. To solve Equation (24), at each time increment the variables $(\boldsymbol{v}_n, \varphi_n)$ are treated simultaneously through a Newton method and, for this reason, we use in what follows the notation $\boldsymbol{x}_n = (\boldsymbol{v}_n, \varphi_n)$.

Inside the loop of the increment time indexed by n, the algorithm we use can be developed in three steps which are the following.

- **For** $n = 0$ **until** N, let $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ be given, φ_0 is calculated by solving the second equation in Problem P_V^{hk} for $n = 0$.

- **A prediction step**: This step provides the initial values $\varphi_{n+1}^0,\ \boldsymbol{u}_{n+1}^0$ and $\boldsymbol{v}_{n+1}^0$ by the formulas: $\varphi_{n+1}^0 = \varphi_n,\ \boldsymbol{u}_{n+1}^0 = \boldsymbol{u}_n$ and $\boldsymbol{v}_{n+1}^0 = \boldsymbol{0}$.
- **A Newton linearization step**: for $i = 0$ until convergence, compute

$$\mathbf{x}_{n+1}^{i+1} = \mathbf{x}_{n+1}^{i} - \left(\frac{P_{n+1}^i}{k} + Q_{n+1}^i + kK_{n+1}^i + T_{n+1}^i\right)^{-1} R\left(\frac{\boldsymbol{v}_{n+1}^i - \boldsymbol{v}_n^i}{k}, \boldsymbol{v}_{n+1}^i, \boldsymbol{u}_{n+1}^i, \varphi_{n+1}^i\right) \qquad (29)$$

where $\boldsymbol{x}_{n+1}^{i+1}$ denotes the pair $(\boldsymbol{v}_{n+1}^{i+1}, \varphi_{n+1}^{i+1})$; i and n represent respectively the Newton iteration index and the time index; $P_{n+1}^i = D_v M(\delta \boldsymbol{v}_{n+1}^i)$ denotes the mass matrix, $Q_{n+1}^i = D_v A(\boldsymbol{v}_{n+1}^i)$ is the damping matrix, $K_{n+1}^i = D_{\boldsymbol{v},\varphi} G(\boldsymbol{u}_{n+1}^i, \varphi_{n+1}^i)$ represents the elastic matrix and $T_{n+1}^i = D_{\boldsymbol{v},\varphi} F(\boldsymbol{v}_{n+1}^i, \varphi_{n+1}^i)$ is the contact tangent matrix; also, $D_{\boldsymbol{v}} M$, $D_{\boldsymbol{v}} A$, $D_{\boldsymbol{v},\varphi} G$ and $D_{\boldsymbol{v},\varphi} F$ denote the differentials of the functions M, A, G and F with respect to the variables $\boldsymbol{v}$ and φ. This leads us to solve the resulting linear system

$$\left(\frac{P_{n+1}^i}{k} + Q_{n+1}^i + kK_{n+1}^i + T_{n+1}^i\right)\Delta \boldsymbol{x}^i = -R\left(\frac{\boldsymbol{v}_{n+1}^i - \boldsymbol{v}_n^i}{k}, \boldsymbol{v}_{n+1}^i, \boldsymbol{u}_{n+1}^i, \varphi_{n+1}^i\right), \qquad (30)$$

where $\Delta \boldsymbol{x}^i = (\Delta \boldsymbol{v}^i, \Delta\varphi^i)$ with $\Delta \boldsymbol{v}^i = \boldsymbol{v}_{n+1}^{i+1} - \boldsymbol{v}_{n+1}^i$ and $\Delta\varphi^i = \varphi_{n+1}^{i+1} - \varphi_{n+1}^i$.

- **A correction step**: Once the system (30) is resolved, we update $\boldsymbol{x}_{n+1}^{i+1}$ and $\boldsymbol{u}_{n+1}^{i+1}$ by $\boldsymbol{x}_{n+1}^{i+1} = \boldsymbol{x}_{n+1}^i + \Delta \boldsymbol{x}^i$ and $\boldsymbol{u}_{n+1}^{i+1} = \boldsymbol{u}_{n+1}^i + k\Delta \boldsymbol{v}^i$.

Note that formulation in Equation (24) has been implemented in the open-source finite element library GetFEM++ (see http://getfem.org/).

4. NUMERICAL SIMULATIONS

Now we illustrate our theoretical results by numerical simulations in the study of two-dimensional test problem. In order to observe the effect of the piezoelectric properties of the material, a physical setting as the one depicted in Figure 1 is considered. Where, $\Omega = (0, 0.03) \times (0, 0.01)$ and $\Gamma_D = \{0\} \times [0, 0.01]$, $\Gamma_N = ([0, 0.03] \times \{0.01\}) \cup (\{0.03\} \times [0, 0.01])$, $\Gamma_C = [0, 0.03] \times \{0\}$, $\Gamma_a = [0, 0.03] \times \{0.01\}$, $\Gamma_b = (\{0\} \times [0, 0.01]) \cup (\{0.03\} \times [0, 0.01])$. On Γ_D the body is clamped, and it is subject to the action of surface pression $\boldsymbol{f}_N(x_1, x_2, t) = (0, -t)N/m$ which acts on the top of the bridge, i.e. on $\Gamma_a = [0, 0.03] \times \{0.01\}$ and the electric potential is free there. The body is in contact with a conductive foundation on its lower boundary Γ_C and we choose Equation (12) as normal damped response contact function, associated with the friction law in Equation (13). Here, we use as material the visco-elasto-piezoelectric body whose constants are taken as [2, 5]. The following data have been used in the numerical simulations:

$$c_\nu = 10^{12}\,\mathrm{Ns/m^2}, \quad c_\tau = 0.3\,\mathrm{Ns/m^2}, \quad k_e = 1, \quad \varphi_f = -48\,\mathrm{V}.$$
$$\boldsymbol{f}_0 = 0\,\mathrm{N/m^2}, \quad q_0 = 0\,\mathrm{C/m^2}, \quad q_b = 0\,\mathrm{C/m}, \quad \rho = 7800\ \mathrm{kg/m^2}.$$
$$T = 0.2\,\mathrm{s}, \quad \boldsymbol{u}_0 = 0\,\mathrm{m}, \quad \boldsymbol{v}_0 = 0\ \mathrm{m/s}.$$

Our interest in this example is to study the influence of the electrical conductivity of the foundation on the contact process and, to this end, we consider the problem both in the case when the foundation is insulated and in the case when it is conductive. When the foundation is insulated there are no electric charges on Γ_C (i.e. $\boldsymbol{D} \cdot \boldsymbol{\nu} = 0$ on Γ_C) and when the foundation is conductive, the normal component of the electric displacement field is assumed to be proportional to the difference between the potential of the foundation and body's surface potential (i.e. $\boldsymbol{D}.\boldsymbol{\nu} = k_e(\varphi - \varphi_f)$ on Γ_C). Our results are shown in Figures 2-5.

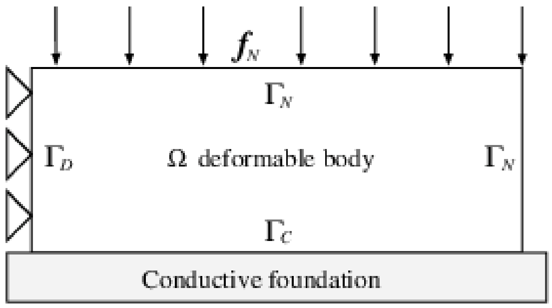
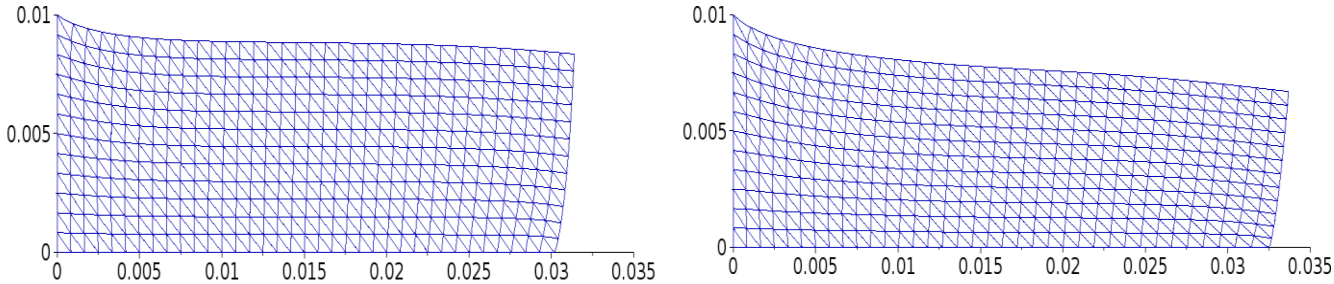


Figure 1. Contact problem with a conductive foundation



(a) (b)

Figure 2. Amplified deformed mesh for (a) an insulated foundation, (b) a conductive foundation

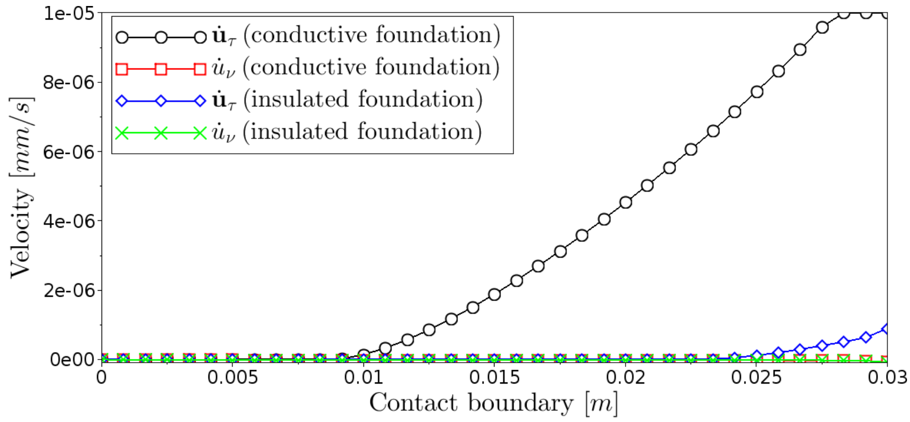
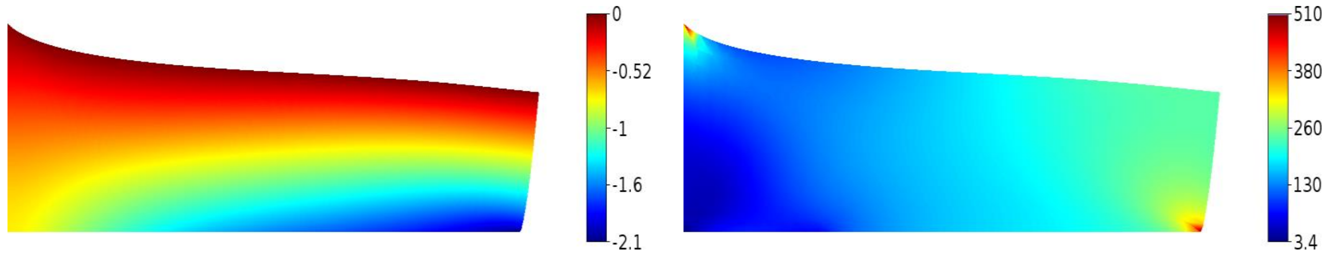


Figure 3. Velocity on Γ_C at final time



(a) (b)

Figure 4. (a) Electric potential [V], (b) the Von Mises stress norm [Pa] at final time

Figure 2 presents the deformed configurations for the two previously mentioned cases, at final time T. Note that in the case of an insulated foundation, the body is compressed by the actions of tractions. However, in the case of a conductive foundation, the shape of the body changes greatly because of the difference in the electrical potential. Such a phenomena is particularly visible on the contact boundary: while there are only 8 nodes in slip status in the first case, there are 25 in the second case (see Figure 3). The values of the computed electric potential and the Von Mises norm of the stress field in the deformed configuration are plotted in Figure 4 for the value $\varphi_f = -48$ V. We note that the magnitude of the electric potential field is large in the zone of contact where the norm of the stress field is large.

Figure 5 shows the relationship of displacements in X, and Y directions with voltages. It shows that the displacement along the X-axis reached 0.8×10^{-3} µm, which is slightly shorter than the displacement along the Y-axis with 1.1×10^{-3} µm for a voltage of 0.75 V.

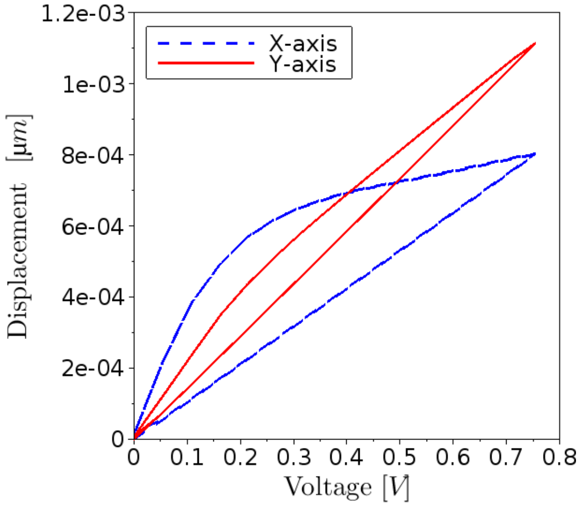


Figure 5. The relationship between voltage and displacement

5. CONCLUSION

A new model of the contact process between an electro-viscoelastic body and the foundation is numerically studied in this paper. The novelties arise in the fact that the process is dynamic, the material behavior is described by an electro-viscoelastic constitutive law and the foundation is electrically conductive. A fully discrete scheme was used to approach the problem and a numerical algorithm which combine the penalty approach with the Newton method was implemented. Moreover, numerical simulations for a representative two-dimensional example were provided in the two different cases insulating and conducting electrical boundary conditions. These simulations describe the inverse piezoelectric effect, i.e. the appearance of strain or stress in the body, due to the action of the electric flied. Also, they underline the effects of the electrical conductivity of the foundation on the process. Performing these simulations, we found that the numerical solution worked well and the convergence was rapid. This work opens the way to study further problems with other conditions for thermally-electrically conductive foundation.

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